Part I

N.B. Each question in part I carries 2 marks.

1. The number of subsets of the set $\{1, 2, \cdots, 10\}$ containing at least one odd integer is

   (a) 2^{10} (b) 2^5 (c) ${}^{10}C_5$ (d) $2^{10} - 2^5$.

   Solution : (d)

   Total number of subsets of the set $\{1, 2, \cdots, 10\}$ is 2^{10}. The number of subsets of the set $\{1, 2, \cdots, 10\}$ containing only even integers is 2^5. Thus the required number is $2^{10} - 2^5$.

2. $1^2 - 2^2 + 3^2 - 4^2 + ... + (2009)^2 - (2010)^2$ is equal to

   (a) zero

   (b) -2021055

   (c) -2019045

   (d) -1010555.

   Solution : (b)

   $1^2 - 2^2 + 3^2 - 4^2 + ... + (2009)^2 - (2010)^2$

   $= (1^2 - 2^2) + (3^2 - 4^2) + ... + ((2009)^2 - (2010)^2)$

   $= (1 - 2)(1 + 2) + (3 - 4)(3 + 4) + ... + (2009 - 2010)(2009 + 2010)$

   $= (-1)(1 + 2 + 3 + 4 + ... + 2009 + 2010)$

   $= -\dfrac{(2010)(2011)}{2} = -2021055.$

3. The coefficient of $(x-1)^3$ in the Taylor expansion of $(x-1)^3 \cos(\pi x)$ about $x = 1$ is

   (a) -1 (b) 1 (c) 6 (d) -6.

   Solution : (a)

   The coefficient of $(x-1)^3$ in the Taylor expansion of $(x-1)^3 \cos(\pi x)$ about $x = 1$ is nothing but the constant term in the Taylor expansion of $\cos(\pi x)$ about $x = 1$. This constant term is $\cos \pi = -1$.

4. The number of non-zero solutions of $z^2 + 2\overline{z} = 0$ is

   (a) 2 (b) 3 (c) 4 (d) 5.

   Solution : (b)

   We have $|z^2| = |-2\overline{z}| = 2|z|$. Suppose $z \neq 0$. Then $|z|^2 = 4 = z\overline{z}$. Hence the equation becomes $z^2 + 2\dfrac{4}{z} = 0$ i.e. $z^3 + 8 = 0$. Hence there are 3 non zero solutions.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 5x - 6 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then f is continuous at
(a) no real number (b) -2 and -3 (c) all rationals (d) 2 and 3.
Solution : (d)
Let a be any real number. Suppose f is continuous at a. Let $\{a_n\}$ and $\{b_n\}$ be sequences of rationals and irrationals respectively converging to a. Then by continuity of f at a,
$\lim f(a_n) = \lim f(b_n)$
$\lim(a_n^2) = \lim(5b_n - 6)$. Therefore $a^2 = 5a - 6$. Hence $a = 2, 3$. Further it is clear that f is continuous at $2, 3$.

6. The number of negative solutions of the equation $e^x - \sin x = 0$ is
(a) 1 (b) 2 (c) 0 (d) infinite.
Solution : (d)
Graphs of e^x and $\sin x$ intersect infinitely many times for negative real numbers.

7. Let $a_n = \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$ and $b_n = \left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n$, and let T_n denote the area of the triangle with sides $a_n - 1, a_n, a_n + 1$. Then
(a) a_n, b_n, T_n are all integers for each $n \in \mathbb{N}$.
(b) a_n, T_n are all integers for each $n \in \mathbb{N}$.
(c) T_n is not an integer for each $n \in \mathbb{N}$.
(d) a_n is an integer for even n and b_n is an integer for odd n.
Solution : (b)
Here we arrive at the answer by the process of elimination ! For $n = 1$, $a_1 = 4, b_1 = 2\sqrt{3}$ and $T_1 = 6$. Therefore (a),(c) and (d) are ruled out.

8. Let $A = (a_{ij})$ be an $m \times n$ matrix where

$$a_{ij} = \begin{cases} 0 & \text{if } i+j \text{ is even} \\ 1 & \text{if } i+j \text{ is odd.} \end{cases}$$

Then the rank of $A = (a_{ij})$ is
(a) m (b) n (c) 2 (d) 3.
Solution : (c)
Observe that alternate rows are identical.

9. The last two digits of 17^{400} are
(a) 17 (b) 09 (c) 01 (d) 89.

Solution : (c)
By Euler's theorem $17^{40} \equiv 1(\text{mod}100)$. Therefore $17^{400} \equiv 1(\text{mod}100)$.

10. The number of values of a for which the equation $x^3 - x + a = 0$ has a double root is
(a) 0 (b) 1 (c) 2 (d) infinite.
Solution : (c)
Observe that the equation $f(x) = 0$ has a double root if and only if it is a common root of $f(x) = 0$ and $f'(x) = 0$. Now $f'(x) = 3x^2 - 1 = 0$ at $x = \pm\dfrac{1}{\sqrt{3}}$. For these values of x, we have two different values of a.

Part II

N.B. Each question in part II carries 5 marks.

1. Find the remainder when $f(x^3)$ is divided by $f(x)$ where
$f(x) = 1 + x + x^2$.
Solution : Let $f(x) = 1+x+x^2$. Then $(x-1)f(x) = x^3-1$. Therefore $f(x)|(x^3 - 1)$.
Now $f(x^3) = 1 + x^3 + x^6 = 3 - 2 + x^3 + x^6 = (x^3 - 1) + (x^6 - 1) + 3$. Therefore $f(x)|[f(x^3) - 3]$. Hence the required remainder is 3, since f being a polynomial of degree 2, the unique remainder is of the form $ax + b$.

2. Let $f : [0, 1] \to [0, 1]$ be a function defined as follows:
$f(1) = 1$, and if $a = 0.a_1a_2a_3....$ is the decimal representation of a (which does not end with a chain of 9's), then $f(a) = 0.0a_10a_20a_3.....$
Discuss the continuity of f at 0.392.
Solution : Let $a = 0.392$. Thus $f(a) = 0.030902$. We shall prove that the given function is not continuous at a. We construct a sequence x_n converging to a such that $f(x_n)$ does not converge to $f(a)$. Let $x_n = 0.39199\ldots9$, where 9 occurs n times at the end. Then $x_n \to a$. Now $f(x_n) = 0.03090109090\ldots09$, where 09 occurs n times at the end. Observe that $f(x_n)$ does not converge to $f(a) = 0.030902$.

3. A bubble chamber contains 3 types of particles. 100 of type x, 200 of type y and 300 of type z. Whenever x and y particles collide they both become z particles, likewise when y and z collide they both become x, and when x and z collide they both become y.
Can the particles in the chamber evolve so that there remain particles of only one type?
Solution : Suppose r, s, t are the number of x, y and z particles in the beginning. When x and y collide we get $(r - 1, s - 1, t + 2)$ as the new numbers say (r', s', t'). We notice that
$t - r \equiv t' - r' \,(\text{mod } 3)$.

Same is true about $r-s$ and $s-t$.
Now in the beginning $r=100, s=200, t=300$ Hence $r-s\equiv 2$ (mod 3).
$s-t\equiv 2$(mod 3).
$t-r\equiv 2$(mod 3).
Thus at any stage $r\not\equiv s$(mod 3),$s\not\equiv t$(mod 3),$t\not\equiv r$(mod 3). But if the particles end up in only one type, then two of r,s,t become zero, say $\overline{r}=\overline{s}=0$. Then certainly $\overline{r}\equiv\overline{s}$(mod3). This is not possible.Hence it is impossible for the particles to end up in one type only.

4. Let $f,g : \mathbb{N}\to\mathbb{N}$ be functions such that f is onto, g is one-one and $f(n)\geq g(n)$ for all $n\in\mathbb{N}$. Prove that $f=g$.
Solution : We shall prove the statement $P(k)$: For every $k\in\mathbf{N}$, there exists a unique $x_k\in\mathbf{N}$ such that $f(x_k)=g(x_k)=k$ by induction on k. Given that f is onto. Thus there exists $x_1\in\mathbf{N}$ such that $f(x_1)=1$. But $g\leq f$, so $g(x_1)=1$. Since g is one-one this x_1 is unique. Thus we have proved $P(1)$. Now let $P(k)$ be true. We shall prove that $P(k+1)$ is true. As f is onto, there exists $x_{k+1}\in\mathbf{N}$ such that $f(x_{k+1})=k+1$. But $g\leq f$ and by induction hypothesis g already takes all values less that $k+1$. So $g(x_{k+1})=k+1$. Since g is one-one this x_{k+1} is unique. Thus by the principle of mathematical induction, the $P(k)$ holds for all natural numbers k. Observe that $P(k)$ implies $f=g$.

Part III

N.B. Each question in part III carries 12 marks.

1. Abhi and Ash play the following game:
A blank 2010×2010 array is taken. Abhi starts the game by writing a real number in any one of the squares of the array. Then Ash writes a real number in any blank square of the array. The game is continued till all the squares are filled with numbers. Abhi wins the game if the determinant of the resulting matrix is non-zero and Ash wins the game if the determinant of the resulting matrix is 0.
Show that Ash can always win the game.
Solution : First note that 2010^2 is an even number. So, since Abhi starts the game, Ash always enters the last number in the array. Hence she tries to make two rows identical. In particular first two.

2. Show that the polynomial equation with real coefficients
$a_nx^n+a_{n-1}x^{n-1}+.....+a_3x^3+x^2+x+1=0$ cannot have all real roots.
Solution : Let $p(x)=a_nx^n+a_{n-1}x^{n-1}+.....+a_3x^3+x^2+x+1$. Note that $p(0)\neq 0$. Thus it is sufficient to prove that $q(x)=p(\frac{1}{x})=0$ cannot have all real roots. Now ,

$$q(x)=x^n+x^{n-1}+x^{n-2}+\cdots+a_n$$

Let $b_1, b_2, \ldots, b_n$ be the roots of $q(x) = 0$. Then $\sum_{i=1}^{n} b_i = -1$ and $\sum b_i b_j = 1$. Thus

$$\sum b_i^2 = (\sum b_i)^2 - 2(\sum b_i b_j) = 1 - 2(1) = -1$$

But if all b_is are real, then $\sum b_i^2 > 0$. Thus all the b_is cannot be real.

3. Find the sum $\displaystyle\sum_{j=0}^{n} \sum_{i=j}^{n} {}^nC_i \, {}^iC_j$.
   Solution : $\sum_{j=0}^{n} \sum_{i=j}^{n} \binom{n}{i}\binom{i}{j}$
   $= \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}$
   $= \sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j}$
   $= \sum_{i=0}^{n} \binom{n}{i} 2^i$
   $= 3^n$.

4. Find the g.c.d. of the numbers $\{2^{13}-2, 3^{13}-3, 4^{13}-4, \cdots, 13^{13}-13\}$.
   Solution : Let d be gcd of the numbers $\{2^{13}-2, 3^{13}-3, 4^{13}-4, \cdots, 13^{13}-13\}$. So $d|(2^{13}-2) = 2 \times 5 \times 7 \times 9 \times 13$.
   $2|(n^{13}-n)$ for all n from 1 to 13.
   $n^2 \equiv 1 (\text{mod } 3) \implies n^{12} \equiv 1 (\text{mod } 3) \implies n^{13} \equiv n (\text{mod } 3) \implies 3|(n^{13}-n)$ for all n from 1 to 13.
   $n^4 \equiv 1 (\text{mod } 5) \implies n^{12} \equiv 1 (\text{mod } 5) \implies n^{13} \equiv n (\text{mod } 5) \implies 5|(n^{13}-n)$ for all n from 1 to 13.
   $n^6 \equiv 1 (\text{mod } 7) \implies n^{12} \equiv 1 (\text{mod } 7) \implies n^{13} \equiv n (\text{mod } 7) \implies 7|(n^{13}-n)$ for all n from 1 to 13.
   $n^{12} \equiv 1 (\text{mod } 13) \implies n^{12} \equiv 1 (\text{mod } 1)3 \implies n^{13} \equiv n (\text{mod } 1)3 \implies 13|(n^{13}-n)$ for all n from 1 to 13.
   Note that 9 does not divide $3^{13}-3$. Hence gcd $\{2^{13}-2, 3^{13}-3, 4^{13}-4, \cdots, 13^{13}-13\}$ is $2 \times 5 \times 7 \times 3 \times 13$.

5. Let $\{a_n\}$ be a sequence of real numbers. Suppose $\{a_{sn}\}$ converges for every fixed positive integer $s > 1$.
   1) If $a_{sn} \to a$ and $a_{tn} \to b$ for some fixed positive integers s and t, then is $a = b$? Justify.
   2) Is the sequence $\{a_n\}$ convergent? Justify.
   Solution : 1) Suppose $a_{sn} \to a$ and $a_{tn} \to b$ for some fixed positive integers s and t. Consider a subsequence $\{a_{stn}\}$. As it is a subsequence of $\{a_{sn}\}$, it converges to a. Also it is a subsequence of $\{a_{tn}\}$, therefore it converges to b. As limit is unique, $a = b$.
   2) The sequence $\{a_n\}$ need not be convergent.
   Define $a_n = 0$ if n is not prime and $a_n = 1$ if n is a prime. This sequence satisfies the above condition but it is not convergent.
