MADHAVA MATHEMATICS COMPETITION, January 6, 2019
Solutions and scheme of marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are
   A) $k \leq 0$ B) $k \geq -4$ C) $k \geq 0$ or $k \leq -4$ D) $-4 \leq k \leq 0$.
   Answer: C
   Equating $kx = (x-1)^2$, we get $x^2 - (k+2)x + 1 = 0$. The discriminant is $(k+2)^2 - 4 = k(k+4) \geq 0$ for $k \geq 0$ or $k \leq -4$.

2. Let $M_2(\mathbb{Z}_2)$ denote the set of all 2×2 matrices with entries from $\mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the set of integers modulo 2. The function $f : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ given by $f(x) = x^2$ is
   A) injective but not surjective B) bijective
   C) surjective but not injective D) neither injective nor surjective.
   Answer: D
   The product $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ shows that the map is not injective and hence not surjective, since the domain and codomain are finite sets having same number of elements.

3. Consider the sequence $4, 0, 4.1, 0, 4.11, 0, 4.111, 0, \cdots$. This sequence
   A) converges to $4\frac{1}{9}$ B) has no convergent subsequence
   C) is unbounded D) is not convergent and has supremum $4\frac{1}{9}$.
   Answer: D
   $a_{2n} = 0 \ \ \forall n \in \mathbb{N}$. This is a convergent subsequence.(Option B not true)
   a_{2n+1} is an increasing subsequence of positive real numbers. Also a_{2n+1} is bounded above by 5. Hence a_{2n+1} is convergent and converges to its supremum,
   supremum of $a_{2n+1} = 4 + \sum_{n=1}^{\infty} \frac{1}{10^n} = 4\frac{1}{9}$.
   Note that $\lim_{n\to\infty} a_{2n+1} = 4\frac{1}{9} \neq \lim_{n\to\infty} a_{2n} = 0$. Thus sequence is bounded (Option C not true) but it is not convergent (Option A incorrect). Clearly Option D is correct.

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x-2)|(x-2)(x-3)|$. The function f is
   A) differentiable at $x = 2$ but not at $x = 3$ B) differentiable at $x = 3$ but not at $x = 2$
   C) differentiable at $x = 2$ and $x = 3$ D) neither differentiable at $x = 2$ nor at $x = 3$.
   Answer: A
   Using definition we get
   $\lim_{x\to 2^+} \frac{f(x) - f(2)}{x-2} = \lim_{x\to 2^-} \frac{f(x) - f(2)}{x-2}$ and $\lim_{x\to 3^+} \frac{f(x) - f(3)}{x-3} \neq \lim_{x\to 3^-} \frac{f(x) - f(3)}{x-3}$.
   Thus the given function is differentiable at 2 but not differentiable at 3

5. The equation $z^2 + \bar{z}^2 = 2$ represents the
   A) parabola B) pair of lines C) hyperbola D) ellipse.
   Answer: C
   The given equation reduces to $x^2 - y^2 = 1$, which represents a hyperbola.

6. The differential equation of the family of parabolas having their vertices at the origin and their foci on the X-axis is
   A) $2xdy - ydx = 0$ B) $xdy + ydx = 0$ C) $2ydx - xdy = 0$ D) $dy - xdx = 0$.
   Answer: A

Let $y^2 = ax$. Differentiating both sides we get, $2y\frac{dy}{dx} = a$. Substituting value of a, we get $2xdy - ydx = 0$.

7. The number of solutions of the equation $\sqrt{1 - \sin x} = \cos x$ in $[0, 5\pi]$ is equal to
A) 3 B) 6 C) 8 D) 11.
Answer: B
Squaring both sides we get $1 - \sin x = 1 - \sin^2 x$ and thus $\sin x = 0$ or $\sin x = 1$. In $[0, 5\pi]$, we then have $x = 0, \pi, 2\pi, 3\pi, 4\pi, \pi/2, 5\pi/2, 7\pi/2$. We need to reject the values $\pi, 3\pi, 5\pi$ as in these cases LHS = 1 and RHS = -1. Hence number of solutions is 6.

8. Consider $\triangle ABC$. Take 3 points on AB, 4 on BC and 5 on CA such that none of the points are vertices of $\triangle ABC$. The number of triangles that can be constructed using these points is
A) 60 B) 205 C) 145 D) 120.
Answer: B
The triangles can be formed in the following ways
1) By choosing one point each on AB, BC, CA. The number of such triangles is 60.
2a) By choosing one point on AB and two on either BC or CA.
This can be done in $3\left[\binom{4}{2} + \binom{5}{2}\right]$.
2b) By choosing one point on BC and two on either AB or CA.
This can be done in $4\left[\binom{3}{2} + \binom{5}{2}\right]$.
2c) By choosing one point on AC and two on either AB or BC.
This can be done in $5\left[\binom{3}{2} + \binom{4}{2}\right]$.
Hence the total number of triangles that can be constructed using these points is 205.

9. The number of primes p such that $p, p + 10, p + 14$ are all prime numbers is
A) 0 B) 1 C) 3 D) infinitely many.
Answer: B
If $p = 3$, we note that 3,13,17 are all prime. Thus $p = 3$ is a solution. Any other prime is either of the type $3k + 1$ or $3k + 2$. If $p = 3k + 1$ for some integer $k \geq 0$, then $p + 14 = 3(k + 5)$ is not a prime. If $p = 3k + 2$ for some integer $k \geq 0$, then $p + 10 = 3(k + 4)$ is not a prime. Thus $p = 3$ is the only solution.

10. A relation R is defined on the set of positive integers as xRy if $2x + y \leq 5$. The relation R is
A) reflexive B) symmetric C) transitive D) None of these.
Answer: D
$(2, 2) \notin R$, so relation R is not reflexive.
Since $(1, 3) \in R$ but $(3, 1) \notin R$, so relation R is not symmetric.
Since $(2, 1) \in R$ and $(1, 3) \in R$ but $(2, 3) \notin R$, so relation R is not transitive.

Part II

N.B. Each question in Part II carries 6 marks.

1. Find all polynomials $p(x)$ such that $(p(x))^2 = 1 + xp(x + 1)$ for all real numbers x.
Solution: Suppose $p(x)$ is a polynomial of degree n and it satisfies
$(p(x))^2 = 1 + xp(x + 1)$ for all real numbers x. Now, the degree of $(p(x))^2$ is $2n$ and the degree of $1 + xp(x + 1)$ is $n + 1$. Comparing left and right hand side of the given equation, we get $2n = n + 1$. Therefore $n = 1$. Hence $p(x)$ is a linear polynomial. [3]
Let $p(x) = ax + b$, where $a \neq 0$. Substituting in the given equation, we get
$(ax + b)^2 = 1 + x(a(x + 1) + b)$. Simplifying this we get a quadratic equation,
$(a^2 - a)x^2 + (2ab - a - b)x + (b^2 - 1) = 0$. This is true for all real numbers x. Hence

$a^2 - a = 0, 2ab - a - b = 0, b^2 - 1 = 0$. This implies that $a = 1, b = 1$. Therefore $p(x) = x + 1$. [3]

2. A transposition of a vector X of length n is created by switching exactly two distinct entries of a vector X. For example, $(1,3,2,4)$ is a transposition of the vector $(1,2,3,4)$ of length 4. Find a vector X if it is given that all the vectors below are transpositions of X: $S = (0,1,1,1,0,0,0,1), T = (1,0,1,1,1,0,0,0), U = (1,0,1,0,1,0,0,1)$, $V = (1,1,1,1,0,0,0,0), W = (1,0,0,1,0,0,1,1)$.
Solution: Let $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$. The first observation is that X should have exactly 4 ones and 4 zeroes as all of S, T, U, V, W are of the same type. Then, the dot product of X with itself will count the number of ones occuring in the vector X. Hence, $\sum_{i=1}^{8} x_i^2 = 4$. Also, if we let Y denote the vector consisting of all ones, then $X \cdot Y = \sum_{i=1}^{8} x_i$ counts the number of ones present in the vector X and hence, $X \cdot Y = 4 = \sum_{i=1}^{8} x_i$. Now note that any permutation of X exchanges one zero and one one. Hence, the dot product of X with any of its transpositions has one less one i.e., $X \cdot S = X \cdot T = X \cdot U = X \cdot V = X \cdot W = 3$. This gives rise to the following system of equations:

$$x_2 + x_3 + x_4 + x_8 = 3,$$
$$x_1 + x_3 + x_4 + x_5 = 3,$$
$$x_1 + x_3 + x_5 + x_8 = 3,$$
$$x_1 + x_2 + x_3 + x_4 = 3,$$
$$x_1 + x_4 + x_7 + x_8 = 3.$$

[3]

Solving the above system, it is easy to check that we get $x_1 = x_4 = x_8$ and $x_2 = x_5$. These equations together with $x_1 + x_4 + x_7 + x_8 = 3$ give that $x_7 = 0$ and $x_1 = x_4 = x_8 = 1$. Since $4 = \sum_{i=1}^{8} x_i$, we get $x_2 + x_3 + x_5 + x_6 + x_7 = 1$. Now if both $x_2 = x_5$ are one, substituting in the above equation will give a contradiction. Hence, $x_2 = x_5 = 0$. From this it is easy to deduce that $x_6 = x_7 = 0, x_3 = 1$. Thus, we get $X = (1,0,1,1,0,0,0,1)$.

[3]

3. In the complex plane, let u, v be two distinct solutions of $z^{2019} - 1 = 0$. Find the probability that $|u + v| \geq 1$.
Solution: Let u, v be two distinct solutions of $z^n - 1 = 0$. Then we can write $u = e^{i2\pi k/n}, v = e^{i2\pi m/n}, m \neq k$.
Observe that $|u + v| = |u||1 + \dfrac{v}{u}| = |1 + e^{i2\pi(m-k)/n}|$
$= |1 + \cos(2\pi(m-k)/n) + i\sin(2\pi(m-k)/n)| = \sqrt{2 + 2\cos(2\pi(m-k)/n)}$
Now $|u + v| \geq 1$ if and only if $\dfrac{-1}{2} \leq \cos\dfrac{2\pi(m-k)}{n}$. [3]
This is true if and only if $\dfrac{-2\pi}{3} \leq \dfrac{2\pi(m-k)}{n} \leq \dfrac{2\pi}{3}$.
This is true if and only if $\dfrac{-1}{3} \leq \dfrac{m-k}{n} \leq \dfrac{1}{3}$.

For $n = 2019$, there are 2×673 possibilities of m for each k. Hence the required probability is $\dfrac{2 \times 673 \times 2019}{2019 \times 2018} = \dfrac{2 \times 673}{2018}$. [3]

4. Let $f : [a,b] \to [a,b]$ be a continuous function which is differentiable on (a,b) and $f(a) = a, f(b) = b$. Prove that there exist two distinct points x_1 and x_2 in (a,b) such that $f'(x_1)f'(x_2) = 1$.
Solution: Let $g = f \circ f$. Then $g(a) = a, g(b) = b$. By Lagrange's Mean Value Theorem,

there exists $c \in (a,b)$ such that $\dfrac{g(b)-g(a)}{b-a} = 1 = g'(c)$.
This implies that $f'(f(c))f'(c) = 1$. [3]
Case 1: If $f(c) \neq c$, then choose $x_1 = c$ and $x_2 = f(c)$. [1]
Case 2: Let $f(c) = c$. Applying Lagrange's Mean Value Theorem to f on intervals $[a,c]$ and $[c,b]$, we get $x_1 \in (a,c)$ and $x_2 \in (c,b)$ such that

$$\frac{f(c)-f(a)}{c-a} = 1 = f'(x_1), \quad \frac{f(b)-f(c)}{b-c} = 1 = f'(x_2).$$

Hence $f'(x_1)f'(x_2) = 1$. [2]

5. Prove that there do not exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^{2018}$ and $g(f(x)) = x^{2019}$.
Solution: Suppose there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^{2018}$ and $g(f(x)) = x^{2019}$. Therefore $(f \circ g)(f(x)) = (f(x))^{2018}$. Since the composition is associative, we have $f \circ (g \circ f)(x) = (f(x))^{2018}$. This implies that

$$f(x^{2019}) = (f(x))^{2018}.$$

[3]
Since 2019 is an odd number, $g \circ f$ is an injective function. Therefore f is also injective. Substituting $x = 0, 1, -1$ in the above equation, we get

$$(f(0))^{2018} = f(0), (f(1))^{2018} = f(1), (f(-1))^{2018} = f(-1).$$

But, only real solutions of $x^{2018} = x$ are 0 and 1. This implies that at least two of $f(0), f(1), f(-1)$ are same which contradicts the injectivity of the function f. Hence there do not exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^{2018}$ and $g(f(x)) = x^{2019}$.
[3]

Part III

1. Let $f(x) = a_0x^n + \cdots + a_n$ be a non-constant polynomial with real coefficients satisfying

$$f(x)f(2x^2) = f(2x^3 + x)$$

for all real numbers x.

(a) Show that $a_n \neq 0$.
(b) Show that f has no real root.
(c) Find a polynomial f satisfying $f(x)f(2x^2) = f(2x^3 + x)$ for all real numbers x.
[13]

Solution:

(a) Let k be the greatest index such that $a_k \neq 0$. Then the left hand side has a form $f(x)f(2x^2) = a_0^2 2^n x^{3n} + \cdots + a_k^2 2^{n-k} x^{3(n-k)}$ and the right hand side has a form $f(2x^3 + x) = a_0 2^n x^{3n} + \cdots + a_k x^{n-k}$. So we must have

$$a_k^2 2^{n-k} x^{3(n-k)} = a_k x^{n-k}, \forall x \in \mathbb{R},$$

which gives $n = k$, that is $a_n = a_k \neq 0$. [4]

(b) Suppose $x_0 \neq 0$ is a root of $f(x)$. Consider a sequence

$$x_{n+1} = 2x_n^3 + x_n, n \geq 0.$$

Note that if $x_0 > 0$, then $\{x_n\}$ is increasing and if $x_0 < 0$, then $\{x_n\}$ is decreasing. From the assumption of the problem, it follows that if $f(x_0) = 0$ with $x_0 \neq 0$, then $f(x_k) = 0, \forall k$. This shows that a non-constant polynomial of degree n has infinitely many roots, which is impossible. Thus f has no real root. [6]

(c) $f(x) = x^2 + 1$ [3]

2. For a subset $X = \{x_1, x_2, \cdots, x_n\}$ of the set of positive integers, $X + X$ denotes the set $\{x_i + x_j \ : \ i \neq j\}$ and $|X|$ denotes the number of elements in X.

   (a) Find subsets A, B of positive integers such that $|A| = |B| = 4, A \neq B$ and $A + A = B + B$.

   (b) Do there exist subsets A, B of positive integers such that $|A| = |B| = 3, A \neq B$ and $A + A = B + B$?

   (c) Show that if $n = 2^k$, then there exist subsets A, B of positive integers such that $|A| = |B| = n, A \neq B$ and $A + A = B + B$. [13]

Solution:

(a) $A = \{1, 4, 6, 7\}, B = \{2, 3, 5, 8\}$. [3]

(b) Let $A = \{a, b, c\}$ and $B = \{d, e, f\}$. Suppose $a < b < c$ and $d < e < f$. Suppose $A + A = B + B$. Now by the given condition, we have

$$a + b = d + e, b + c = e + f, a + c = d + f.$$

Adding all these we get, $a + b + c = d + e + f$. This implies $a = d, b = e, c = f$. This is contradiction. [4]

(c) Note that if sets $A = \{a_1, \cdots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ have property that $|A| = |B| = n, A \neq B$ and $A+A = B+B$, then for a suitable positive integert m, the sets $A' = \{a_1, \ldots, a_n, b_1+m, \cdots, b_n+m\}$ and $B' = \{a_1+m, \ldots, a_n+m, b_1, \ldots, b_n\}$ will have property that $|A'| = |B'| = 2n, A' \neq B'$ and $A' + A' = B' + B'$. Now starting with $A = \{1, 4\}$ and $B = \{2, 3\}$, for any positive integert k, we can inductively find the sets of sizes 2^k with desired property. [6]

3. On the real line place an object at 1. After every flip of a fair coin, move the object to the right by 1 unit if the outcome is Head and to the left by 1 unit if the outcome is Tail. Let N be a fixed positive integer. Game ends when the object reaches either 0 or N. Let $P(N)$ denote the probability of the object reaching N.

   (a) Find $P(3)$.

   (b) Find the formula for $P(N)$ for any positive integer N. [12]

Solution:

(a) Starting the game at 1, the possible outcomes to reach 3 without reaching zero are HH, HTHH, HTHTHH and so on. Hence the probability of reaching 3 without going to zero is given by a geometric series $\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$ which adds up to $\frac{1}{3}$. Hence $P(3) = \frac{1}{3}$. [6]

(b) For any positive integer N, $P(N) = P(N-1)[P(N-1 \to N)]$ where $[P(N-1 \to N)]$ denotes probabilty of reaching N from $N-1$ without reachin zero. Now by symmetry, $[P(N-1 \to N)]$ is equal to $[P(1 \to 0)]$ without reaching N. But $[P(1 \to 0)]$ is equal to $1 - P(N)$. Thus we have a recurrence relation $P(N) = P(N-1)(1 - P(N))$. Now with $P(1) = 1, P(2) = \frac{1}{2}, P(3) = \frac{1}{3}$, the recurrence relation obtained above allows us to prove that $P(N) = \frac{1}{N}$. [6]

4. Let f be a real valued differentiable function on $(0, \infty)$ satisfying

   (a) $|f(x)| \leq 5$ and

(b) $f(x)f'(x) \geq \sin x$ for all $x \in (0, \infty)$.

Does $\lim_{x \to \infty} f(x)$ exist? [12]

Solution: Consider $F(x) = f^2(x) + 2\cos x$ defined on $(0, \infty)$. Then by (a),

$$|F(x)| \leq |f(x)|^2 + 2|\cos x| \leq 5^2 + 2.$$

By (b), $F'(x) = 2f(x)f'(x) - 2\sin x \geq 0, \ \forall x \in (0, \infty)$ implying that $F(x)$ is an increasing function. [6]

Let $\{x_n\} := \{2\pi, 2\pi + \frac{\pi}{2}, 4\pi, 4\pi + \frac{\pi}{2}, 6\pi, 6\pi + \frac{\pi}{2}, \cdots\}$. Observe that $\forall n, \ x_n > 0, \{x_n\}$ is an increasing sequence and $x_n \to \infty$ as $n \to \infty$.

Put $u_n = F(x_n)$. Now observe that $\{u_n\}$ is an increasing sequence which is bounded above. Thus $\{u_n\}$ is convergent. Assume that $\lim_{x \to \infty} f(x)$ exists. This implies that if $v_n = f(x_n)$, then $\lim_{n \to \infty} v_n$ exists. Therefore $\lim_{n \to \infty} [F(x_n) - f^2(x_n)]$ exists. Now

$$\lim_{n \to \infty} [F(x_n) - f^2(x_n)] = \lim_{n \to \infty} u_n - \lim_{x \to \infty} v_n^2.$$

This implies that $\lim_{n \to \infty} \cos x_n$ exists. This is not possible because the sequence $\{\cos x_n\} = \{1, 0, 1, 0, \cdots\}$ has no limit. [6]
