MADHAVA MATHEMATICS COMPETITION, January 8, 2017
Solutions and scheme of marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The number $\sqrt{2}e^{i\pi}$ is:
A) a rational number.
B) an irrational number.
C) a purely imaginary number.
D) a complex number of the type $a + ib$ where a, b are non-zero real numbers.
Answer: B
The number is $-\sqrt{2}$ using the relation $e^{i\pi} = -1$.

2. Let $P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The rank of P^4 is: A) 1 B) 2 C) 3 D) 4.
Answer: D
Det $P \neq 0, \Rightarrow$ Det $P^4 \neq 0$. Thus rank of P^4 is 4.

3. Let $y_1(x)$ and $y_2(x)$ be the solutions of the differentiable equation $\dfrac{dy}{dx} = y + 17$ with initial conditions $y_1(0) = 0, y_2(0) = 1$. Which of the following statements is true?
A) y_1 and y_2 will never intersect.
B) y_1 and y_2 will intersect at $x = e$.
C) y_1 and y_2 will intersect at $x = 17$.
D) y_1 and y_2 will intersect at $x = 1$.
Answer: A
Solving the differentiable equation we get $y_1 = -17 + 17e^x$ and $y_2 = -17 + 18e^x$. The two curves never intersect.

4. Suppose f and g are differentiable functions and $h(x) = f(x)g(x)$. Let $h(1) = 24$, $g(1) = 6$, $f'(1) = -2$, $h'(1) = 20$. Then the value of $g'(1)$ is
A) 8 B) 4 C) 2 D) 16.
Answer: A
$h(x) = f(x)g(x)$. Thus we get $h^{'}(x) = f^{'}(x)g(x) + f(x)g^{'}(x)$
$h^{'}(1) = f^{'}(1)g(1) + f(1)g^{'}(1)$. Therefore $20 = (-2)(6) + f(1)g^{'}(1)$.
$f(1)g^{'}(1) = 32$. Now $h(1) = f(1)g(1)$. Therefore $24 = 6f(1)$.
Thus $f(1) = 4$ and $g^{'}(1) = 8$.

5. In how many regions is the plane divided when the following equations are graphed, not considering the axes? $y = x^2,\ y = 2^x$
A) 3 B) 4 C) 5 D) 6.
Answer: D
Plot graph of the two functions $y = x^2$ and $y = 2^x$.

6. For $0 \leq x < 2\pi$, the number of solutions of the equation $\sin^2 x + 3\sin x \cos x + 2\cos^2 x = 0$ is
A) 1 B) 2 C) 3 D) 4.
Answer: D
Note $\cos^2(x) \neq 0$. Dividing by $\cos^2(x)$ we get $\tan^2(x) + 3\tan x + 2 = 0$. Thus $\tan x = -1$ or -2. Since $\tan x$ has period π and range of $\tan x$ is $(-\infty, \infty)$, the number of solutions of the given equation in the interval $0 \leq x < 2\pi$ is equal to 4.

7. The minimum value of the function $f(x) = x^x,\ x \in (0, \infty)$ is
A) $\left(\frac{1}{10}\right)^{\frac{1}{10}}$ B) $10^{\frac{1}{10}}$ C) $\frac{1}{e}$ D) $\left(\frac{1}{e}\right)^{\frac{1}{e}}$.
Answer: D
$y = x^x$. Taking log on both sides $\ln y = x \ln x$.
$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$. Solving $\frac{dy}{dx} = 0 \Rightarrow \ln_e x = -1$. Thus minimum value is obtained at $x = 1/e$ and is $\left(\frac{1}{e}\right)^{\frac{1}{e}}$. Note that $\frac{d^2y}{dx^2}(\frac{1}{e}) > 0$.

8. Let f be a twice differentiable function on $\mathbb{R}$. Also $f^{''}(x) > 0$ for all $x \in \mathbb{R}$. Which of the following statements is true?
A) $f(x) = 0$ has exactly two solutions on $\mathbb{R}$.
B) $f(x) = 0$ has a positive solution if $f(0) = 0$ and $f^{'}(0) = 0$.
C) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f^{'}(0) > 0$.
D) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f^{'}(0) < 0$.
Answer: C
$f^{''}(x) > 0 \Rightarrow f^{'}(x)$ is increasing. Also $f^{'}(0) > 0 \Rightarrow f^{'}(x) > 0$ if $x > 0$. $\Rightarrow f(x) = 0$ has no positive solution.

9. If $x^2 + x + 1 = 0$, then the value of $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \ldots + \left(x^{27} + \frac{1}{x^{27}}\right)^2$ is
A) 27 B) 54 C) 0 D) -27.
Answer: B
$$\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \cdots + \left(x^{27} + \frac{1}{x^{27}}\right)^2$$
$$= \left(\frac{x^2+1}{x}\right)^2 + \left(\frac{x^4+1}{x^2}\right)^2 + \cdots + \left(\frac{x^{54}+1}{x^{27}}\right)^2$$
$$= 9((-1)^2 + (-1)^2 + 2^2) = 54.$$

10. Let M= $\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$. Then $M^{2017} =$
A)$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ B)$\begin{pmatrix} -2^{2017} & -1 \\ 3^{2017} & 1 \end{pmatrix}$ C) $\begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$ D) $\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$.
Answer: D
Note $M^3 = I$. Thus $M^{2016} = I$ and $M^{2017} = M$.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let a, b, c be real numbers such that $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $abc = 1$. Prove that at least one of a, b, c is 1.
Solution: Let $\lambda = a + b + c$. Then $\lambda = a + b + c = \frac{bc + ac + ab}{abc} = bc + ac + ab$. **2 marks**
The numbers a, b, c are roots of the polynomial $x^3 - \lambda x^2 + \lambda x - 1$. Observe that $x = 1$ is one of the roots of this polynomial. **4 marks**

2. Let $c_1, c_2, \ldots, c_9$ be the zeros of the polynomial $z^9 - 6z^7 + 12z^6 + 18z^4 - 24z^3 + 30z^2 - z + 2017$. If $S(z) = \sum_{k=1}^{9} |z - c_k|^2$, then prove that $S(z)$ is constant on the circle $|z| = 100$.
Solution: Observe that $\sum_{k=1}^{9} c_k = 0$. **2 marks**

$S(z) = \sum_{k=1}^{9} |z - c_k|^2 = \sum_{k=1}^{9} (z - c_k)\overline{(z - c_k)} = \sum_{k=1}^{9} (z\overline{z} - z\overline{c_k} - c_k\overline{z} + c_k\overline{c_k}) = \sum_{k=1}^{9} |z|^2 + \sum_{k=1}^{9} |c_k|^2 - z\sum_{k=1}^{9} \overline{c_k} - \overline{z}\sum_{k=1}^{9} c_k = \sum_{k=1}^{9} |z|^2 + \sum_{k=1}^{9} |c_k|^2 = \sum_{k=1}^{9} (100)^2 + \sum_{k=1}^{9} |c_k|^2 =$ constant. **4 marks**

3. Let f be a monic polynomial with real coefficients. Let $\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} f\left(\frac{1}{x}\right)$ and $f(x) \geq f(1)$ for all $x \in \mathbb{R}$. Find f.
   Solution: Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Then $f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$ and $f''(x) = 2a_2 + 6a_3x + \cdots + n(n-1)a_nx^{n-2}$. Also $f(\frac{1}{x}) = a_0 + a_1(\frac{1}{x}) + \cdots + a_n(\frac{1}{x})^n$. Now $\lim_{x\to\infty} f\left(\frac{1}{x}\right) = a_0$. Therefore $\lim_{x\to\infty} f''(x) = a_0$. Hence $f''(x) = 2a_2$ and $a_3 = a_4 = \cdots = a_n = 0$. **3 marks**
   Since f is monic, $a_2 = 1$. Now $\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} f\left(\frac{1}{x}\right)$ implies $a_0 = 2a_2 = 2$. Also $f(x) \geq f(1)$ for all $x \in \mathbb{R}$ implies f has minimum at $x = 1$. Therefore $f'(1) = 0$. Hence $a_1 + 2a_2 = 0$. Therefore $a_1 = -2$. Hence $f(x) = x^2 - 2x + 2$. **3 marks**

4. Call a set of integers $non-isolated$ if for every $a \in A$ at least one of the numbers $a - 1$ and $a + 1$ also belongs to A. Prove that the number of 5−element $non-isolated$ subsets of $\{1, 2, \ldots, n\}$ is $(n-4)^2$.
   Solution: Let $\{a_1, a_2, a_3, a_4, a_5\}$ be 5-element $non-isolated$ subset of $\{1, 2, \ldots, n\}$ such that $a_1 < a_2 < a_3 < a_4 < a_5$. Then $a_2 = a_1 + 1, a_4 = a_5 - 1$. Further $a_3 = a_2 + 1$ or $a_3 = a_4 - 1$. Clearly $1 \leq a_1 \leq n - 4$. For each choice of a_1, a_5 has $(n-3) - a_1$ choices. For a_3, there are 2 choices. So total number of such sets is $2[(n-4) + (n-5) + \cdots + 1] = 2\frac{(n-4)(n-3)}{2} = (n-4)(n-3)$. **4 marks**
   But $a_3 = a_2 + 1$ as well as $a_3 = a_4 - 1$ gets counted twice. So total number of such sets is $(n-4)(n-3) - (n-4) = (n-4)(n-4) = (n-4)^2$. **2**

5. Find all positive integers n for which a permutation $a_1, a_2, \ldots, a_n$ of $\{1, 2, \ldots, n\}$ can be found such that $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \ldots + a_n$ leave distinct remainders modulo $n + 1$.
   Solution: Let $a_1, a_2, \ldots, a_n$ be a permutation of $\{1, 2, \ldots, n\}$ such that $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \ldots + a_n$ leave distinct remainders modulo $n + 1$. If n is even, then $a_1 + a_2 + \cdots + a_n = \frac{n(n+1)}{2} \equiv 0(\mod n + 1)$. So remainders modulo $n + 1$ can not be distinct. **3 marks**
   Let n be odd. Then choose $a_1 = 1, a_2 = n - 1, a_3 = 3, a_4 = n - 3$ and so on. Then $a_1 \equiv 1(\mod n + 1), a_1 + a_2 \equiv n(\mod n + 1), a_1 + a_2 + a_3 = n + 3 \equiv 2(\mod n + 1), a_1 + a_2 + a_3 + a_4 = 2n \equiv n - 1(\mod n + 1), \cdots$. This gives required permutation. **3 marks**

Part III

1. Do there exist 100 lines in the plane, no three concurrent such that they intersect exactly in 2017 points? [12]
   Solution: Consider k sets of parallel lines having respectively $m_1, m_2, \cdots, m_k$ lines in each set. Then we need to solve the equations $m_1 + m_2 + \cdots + m_k = 100$ and $\sum_{i<j} m_im_j = 2017$. **4 marks**
   $\left(\sum m_i\right)^2 = \sum m_i^2 + 2\sum_{i<j} m_im_j$

Therefore $\sum m_i^2 = 10,000 - 2(2017) = 5966$. Thus we need to find 100 numbers satisfying $\sum m_i = 100$ and $\sum m_i^2 = 5966$. **3 marks**
We can get any one of the following solutions:

(a) $75, 18, 3, 2, 2$

(b) $77, 3, 2, 2, 2, 2, 1$(12 times)

(c) $77, 4, 2, 1$(17 times) **5 marks**

2. On the parabola $y = x^2$, a sequence of points $P_n(x_n, y_n)$ is selected recursively where the points P_1, P_2 are arbitrarily selected distinct points. Having selected P_n, tangents drawn at P_{n-1} and P_n meet at say Q_n. Suppose P_{n+1} is the point of intersection of $y = x^2$ and the line passing through Q_n parallel to Y-axis. Under what conditions on P_1, P_2

   (a) both the sequences $\{x_n\}$ and $\{y_n\}$ converge?

   (b) $\{x_n\}$ and $\{y_n\}$ both converge to 0? [13]

Solution:

(a) Since $y_n = x_n^2$, it is enough to discuss the convergence of $\{x_n\}$. Tangents at x_n, x_{n-1} are given by $y = x_n^2 + 2x_n(x - x_n) = 2x_nx - x_n^2$ and $y = 2x_{n-1}x - x_{n-1}^2$.
Solving we get, $x_{n+1} = \dfrac{x_{n-1} + x_n}{2}$. Therefore $P_{n+1}\left(\dfrac{x_{n-1} + x_n}{2}, (\dfrac{x_{n-1} + x_n}{2})^2\right)$.
4 marks
Now all x_n are within the interval $[x_1, x_2]$ and all are distinct. Hence $\{x_n\}$ converges. **3 marks**

(b) We first consider a special case where $x_1 = 0, x_2 = 1$. Then the sequence $\{x_n\}$ is $0, 1, \frac{1}{2}, \frac{3}{4}, \frac{1}{2} + \frac{1}{8}, \frac{1}{2} + \frac{1}{8} + \frac{1}{16}, \frac{1}{2} + \frac{1}{8} + \frac{1}{32}, \cdots$. Its sub sequence is $\frac{1}{2}, \frac{1}{2} + \frac{1}{8}, \frac{1}{2} + \frac{1}{8} + \frac{1}{32}, \cdots$ which are partial sums of the geometric series $\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}}$ which converges and sum is given by $\frac{2}{3}$. Thus in this case x_n converges to $\frac{2}{3}$.
Now in general, for any $x_1 < x_2$, we define $z_{n-1} = \dfrac{x_n - x_1}{x_2 - x_1}$. Note that $z_0 = 0$ and $z_1 = 1$ and the sequence z_n satisfies the relation $z_{n+1} = \dfrac{z_{n-1} + z_n}{2}$. Thus by the special case above, z_n converges to $\frac{2}{3}$. Now observe that the limit of x_n is the real number x which divides the interval $[x_1, x_2]$ in the ratio $2 : 1$. For $x = 0$, we need to take $x_1 \neq 0$ and $x_2 = -\frac{1}{2}x_1$. **6 marks**

3. (a) Show that there does not exist a 3-digit number A such that $10^3A + A$ is a perfect square.

   (b) Show that there exists an n-digit ($n > 3$) number A such that $10^nA + A$ is a perfect square. [12]

Solution:

(a) If $10^3A + A = A(1001)$ is a perfect square, then $7|A, 11|A$ and $13|A$. Therefore $A \geq 1001$. That is A has bigger than or equal to 4 digits. **3 marks**

(b) For some n if $10^nA + A = A(10^n + 1)$ is a perfect square, then $10^n + 1$ must be divisible by a square bigger than 1. Because if no perfect square divides $10^n + 1$, then all prime divisors of $10^n + 1$ must appear in factorization of A. This makes $A \geq 10^n + 1$, but $A < 10^n + 1$. Therefore $10^n + 1$ has a square factor. **4 marks**

Now observe that $10^{11}+1 = (11)(10^{10}-10^9+10^8-\cdots-10+1) = (11)(9090909091)$ and thus 121 divides $10^{11}+1$. **2 marks**

Thus we can choose A as $\dfrac{10^{11}+1}{121} \times 9$ so that $10^{10} \leq A < 10^{11}$ and $(10^n+1)A$ is a perfect square. **3 marks**

4. For $n \times n$ matrices A, B, let $C = AB - BA$. If C commutes with both A and B, then

   (a) Show that $AB^k - B^kA = kB^{k-1}C$ for every positive integer k.

   (b) Show that there exists a positive integer m such that $C^m = 0$. [13]

Solution:

(a) The proof is by induction. The result is true for $k = 1, 2$. $AB - BA = C$ and $AB^2 - B^2A = (AB - BA)B + B(AB - BA) = CB + BC = 2BC$. **1 marks**
Assume that the result is true upto $k-1$. $AB^k - B^kA = (AB - BA)B^{k-1} + B(AB^{k-1} - B^{k-1}A) = CB^{k-1} + B(k-1)B^{k-2}C = kB^{k-1}C$. **3 marks**

(b) Hence for any polynomial $q(x)$, $Aq(B) - q(B)A = q'(B)C$, where q' is a derivative of q. In particular, let $p(x)$ be the characteristic polynomial of B. Now by Caley-Hamilton Theorem,
$0 = Ap(B) - p(B)A = p'(B)C$. This proves $p'(B)C = 0$. **4 marks**
$0 = Ap'(B)C - p'(B)AC = (Ap'(B) - p'(B)A)C = p''(B)C^2$. This proves $p''(B)C^2 = 0$. Inductively, we have $p^{(k)}(B)C^k = 0$. Therefore for $k = n$, we have $n!C^n = 0$. Hence $C^n = 0$. **5 marks**