MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune (Autonomous)
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 29/01/2023 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The number of positive divisors of $2^{24} - 1$ is
(A) 192 (B) 48 (C) 96 (D) 24.

2. The equation Re $(z^2) = 0$ represents
(A) a circle (B) a pair of straight lines (C) an ellipse (D) a parabola.

3. If $A = \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$ and $\det A^3 = 125$, then the values of α are
(A) ± 1 (B) ± 2 (C) ± 3 (D) ± 5.

4. Let A, B, C be three non-collinear points in a plane. The number of points at a distance 1 from A, 2 from B and 3 from C is
(A) exactly 1 (B) at most 1 (C) at most 2 (D) always 0.

5. Let $A = \{x \in [-2, 3] : \ \cos x > 0\}$. Then
(A) $\inf A = 0$ (B) $\sup A = \pi$ (C) $\inf A = -\pi/2$ (D) $\sup A = 3$.

6. Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| \leq \dfrac{2023}{n}|a_n - a_{n-1}|, \ \forall n$. Then the sequence $\{a_n\}$ is
(A) not Cauchy (B) Cauchy but not convergent (C) convergent (D) not bounded.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and F be a primitive of f (i.e. $F' = f$). If $3x^2F(x) = f(x)$ for all $x \in \mathbb{R}$ then $f(x) =$
(A) e^{x^3} (B) $3x^2e^{x^3}$ (C) $x^2e^{x^2}$ (D) $3xe^{x^3}$.

8. $1 \times 2 - 2 \times 3 + 3 \times 4 - 4 \times 5 + \cdots - (2022) \times (2023) =$
(A) $(-2)(1011)(1012)$ (B) $-(1011)(1012)$
(C) $(-4)(1011)(1012)$ (D) $2(1011)(1012)$.

9. The number of times the digit 7 is written while listing all integers from 1 to $1,00,000$ is
(A) 10^4 (B) $5(10)^4 - 1$ (C) 10^5 (D) $5(10)^4$.

10. The differential equation $y'^2 - (x + \sin x)y' + x \sin x = 0$, with $y(0) = 0$ has
(A) unique solution (B) two solutions (C) no solution (D) four solutions.

Part II

N.B. Each question in Part II carries 6 marks.

1. Consider $f(x) = x[x^2]$, where $[x^2]$ is the greatest integer less than or equal to x^2. Find the area of the region above X-axis and below $f(x), 1 \leq x \leq 10$.
2. In how many ways can numbers from 1 to 100 be arranged in a circle such that sum of pair of integers placed opposite each other is the same? (arrangements are equivalent up to rotation).
3. Find all triplets (x, y, z) of integers satisfying $x^2 + y^2 + z^2 = 16(x + y + z)$.
4. Suppose A is a singular matrix of order 3 with complex entries all of which having absolute value 1. Show that two rows or two columns of the matrix A are proportional.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f^3(x) = x$. Prove that $f^2(x) = x$.

Part III

1. Find [12]

   (a) $\lim_{n\to\infty} \frac{\gcd(1,6) + \gcd(2,6) + \cdots + \gcd(n,6)}{1 + 2 + \cdots + n}$;

   (b) $\lim_{n\to\infty} \frac{lcm(1,6) + lcm(2,6) + \cdots + lcm(n,6)}{1 + 2 + \cdots + n}$.

2. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 4$. [12]

   (a) Find the value of the determinant of a matrix $A = \begin{pmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{pmatrix}$.

   (b) Find the maximum and minimum value of the above determinant.

3. For every $t \in \mathbb{R}$, let L_t be the line segment joining $(0, 1)$ with $(t, 0)$. Suppose L_t intersects the parabola $y = x^2$ at the point P_t. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(t) = y$-coordinate of P_t. Answer the following questions with justification: [13]

   (a) Is f continuous?

   (b) Is f bounded?

   (c) What is $\lim_{t\to\infty} f(t)$?

   (d) Is f differentiable at 0?

4. The sequence $\{q_n(x)\}$ of polynomials is defined by $q_1(x) = 1 + x, q_2(x) = 1 + 2x$ and for $m \geq 1$ by

   $$q_{2m+1}(x) = q_{2m}(x) + (m + 1)xq_{2m-1}(x),$$

   $$q_{2m+2}(x) = q_{2m+1}(x) + (m + 1)xq_{2m}(x).$$

   Let x_n be the largest real solution of $q_n(x) = 0$. Prove that [13]

   (a) the sequence $\{x_n\}$ is increasing.

   (b) $x_{2m+2} > \frac{-1}{m+1}$ for $m \geq 1$.

   (c) the sequence $\{x_n\}$ converges to 0.