MADHAVA MATHEMATICS COMPETITION

Date: 29/01/2023 Max. Marks: 100

Solutions and Scheme of marking:

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The number of positive divisors of $2^{24}-1$ is
(A) 192 (B) 48 (C) 96 (D) 24
Ans:(C)

2. The equation Re $(z^2)=0$ represents
(A) a circle (B) a pair of straight lines (C) an ellipse (D) a parabola
Ans:(B)

3. If $A=\begin{pmatrix}\alpha & 2\\ 2 & \alpha\end{pmatrix}$ and $\det A^3=125$, then the value of α is
(A) ± 1 (B) ± 2 (C) ± 3 (D) ± 5.
Ans: (C).

4. Let A, B, C be three non-collinear points in a plane. The number of points at a distance 1 from A, 2 from B and 3 from C is
(A) exactly 1 (B) at most 1 (C) at most 2 (D) always 0.
Ans: (B)

5. Let $A=\{x\in[-2,3]: \ \cos x>0\}$. Then
(A) $\inf A=0$ (B) $\sup A=\pi$ (C) $\inf A=-\pi/2$ (D) $\sup A=3$.
Ans: (C)

6. Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1}-a_n|\leq \frac{2023}{n}|a_n-a_{n-1}|, \ \forall n$. Then the sequence $\{a_n\}$ is
(A) not Cauchy (B) Cauchy but not convergent (C) convergent (D) not bounded.
Ans: (C)

7. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function and F is a primitive of f (i.e. $F'=f$). If $3x^2F(x)=f(x)$ for all $x\in\mathbb{R}$ then $f(x)=$
(A) e^{x^3} (B) $3x^2e^{x^3}$ (C) $x^2e^{x^2}$ (D) $3xe^{x^3}$.
Ans: (B)

8. $1\times 2-2\times 3+3\times 4-4\times 5+\cdots-(2022)\times(2023)=$
(A) $(-2)(1011)(1012)$ (B) $-(1011)(1012)$
(C) $(-4)(1011)(1012)$ (D) $2(1011)(1012)$.
Ans: (A)

9. The number of times the digit 7 is written while listing all integers from 1 to $1,00,000$ is
(A) 10^4 (B) $5(10)^4-1$ (C) 10^5 (D) $5(10)^4$.
Ans: (D)

10. The differential equation $y'^2-(x+\sin x)y'+x\sin x=0$, with $y(0)=0$ has
(A) unique solution (B) two solutions (C) no solution (D) four solutions.
Ans: (B)

Part II

N.B. Each question in Part II carries 6 marks.

1. Consider $f(x) = x[x^2]$, where $[x^2]$ is the greatest integer less than or equal to x^2. Find the area of the region above X-axis and below $f(x), 1 \leq x \leq 10$.

Solution:

Observe that

$$f(x) = \begin{cases} x, & 1 \leq x < \sqrt{2} \\ 2x, & \sqrt{2} \leq x < \sqrt{3} \\ \vdots & \vdots \\ 99x, & \sqrt{99} \leq x < 10 \\ 1000 & x = 10 \end{cases}$$ [4 Marks]

Thus, the area of the required region is

$$\sum_{k=1}^{99} \int_{\sqrt{k}}^{\sqrt{k+1}} kx dx = \sum_{k=1}^{99} \frac{k}{2} \left[x^2\right]_{\sqrt{k}}^{\sqrt{k+1}} = \sum_{k=1}^{99} \frac{k}{2} = 99 \times 25 = 2475$$ [2 Marks]

2. In how many ways can numbers from 1 to 100 be arranged in a circle such that sum of two integers placed opposite each other is the same? (arrangements are equivalent up to rotation.)

Solution:

It is clear that opposite pairs must be $(1, 100), (2, 99), \ldots, (50, 51)$ [1 Mark]
1 can be placed anywhere and 100 must be placed opposite to 1. This can be done in exactly one way as all places are identical to start with.
Now, 2 has 98 options and then 99 has to be placed opposite to 2.
3 has 96 options and then 96 must be placed opposite to 3 and so on.
By multiplication principle, the required number of ways is
$98 \times 96 \times 94 \times \cdots \times 2 = 2^{49} \times 49!$. [5 Mark]

3. Find all triplets (x, y, z) of integers satisfying $x^2 + y^2 + z^2 = 16(x + y + z)$.

Solution:

Let (x, y, z) be a triplet satisfying the given condition.
Thus, $x^2 + y^2 + z^2 = 16(x + y + z) \ldots\ldots (*)$
Every square is congruent to $0, 1$ or 4 modulo 8, in fact, odd squares give remainder 1 when divided by 8. Since RHS of (*) is divisible by 8, so must be the LHS.
Hence, x, y, z must be even integers. [3 Marks]
Let $x = 2x_1$, $y = 2y_1$ and $z = 2z_1$, for some $x_1, y_1, z_1 \in \mathbb{Z}$.
Substituting in (*), we get, $x_1^2 + y_1^2 + z_1^2 = 8(x_1 + y_1 + z_1) \ldots\ldots (**)$
By similar argument as above, we must have x_1, y_1, z_1 must be even.
Let $x_1 = 2x_2$, $y_1 = 2y_2$ and $z_1 = 2z_2$, for some $x_2, y_2, z_2 \in \mathbb{Z}$.
Substituting in (**), we get, $x_2^2 + y_2^2 + z_2^2 = 4(x_2 + y_2 + z_2) \ldots\ldots (***)$
Once again, arguing in similar manner, we must have x_2, y_2, z_2 to be even integers.
Let $x_2 = 2x_3$, $y_2 = 2y_3$ and $z_2 = 2z_3$, for some $x_3, y_3, z_3 \in \mathbb{Z}$.
Substituting in (***), we get, $x_3^2 + y_3^2 + z_3^2 - 2(x_3 + y_3 + z_3) = 0$.
That is, $(x_3 - 1)^2 + (y_3 - 1)^2 + (z_3 - 1)^2 = 3$
This implies, each of $|x_3 - 1|, |y_3 - 1|$ and $|z_3 - 1|$ must be 1.
Hence, x_3, y_3, z_3 are either 2 or 0 and thus, x, y, z can be either 16 or 0.
Thus, all possible triplets satisfying given condition are: $(16, 16, 16), (16, 16, 0), (16, 0, 16), (0, 16, 16), (0, 0, 16), (0, 16, 0), (16, 0, 0), (0, 0, 0)$. [3 Marks]

4. Suppose A is a singular matrix of order 3 with complex entries all of which having absolute value 1. Show that two rows or two columns of the matrix A are proportional.

Solution:
By suitable multiplication on each row and column, the matrix A can be written as $k\begin{pmatrix}1 & 1 & 1\\ 1 & a & b\\ 1 & c & d\end{pmatrix}$, where a, b, c, d, k are complex numbers of absolute value 1. [2 Marks]
The relation $\det(A) = 0$ gives $(a-1)(d-1) = (b-1)(c-1)$. Taking the complex conjugates of the above equation and using the fact that each number is of absolute value one, to get $\overline{ad}(a-1)(d-1) = \overline{bc}(b-1)(c-1)$. [2 Marks]
If $(a-1)(d-1) = 0$, then $(b-1)(c-1) = 0$ and two rows or columns are equal to $(1\ \ 1\ \ 1)$ and result is proved.
Suppose $(a-1)(d-1) = (b-1)(c-1) \neq 0$. Then $\overline{ad} = \overline{bc}$ and $ad = bc$. From $(a-1)(d-1) = (b-1)(c-1)$ we get $a+d = b+c$. Hence $\{a, d\} = \{b, c\}$ or $a = b = c = d$. It follows that the bottom two rows or the rightmost two columns are equal. [2 Marks]

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f^3(x) = x$. Prove that $f^2(x) = x$.
Solution:
We first prove that f is injective.
Since, $f(x) = f(y) \implies f^3(x) = f^3(y) \implies x = y$
hence, f is an injective function.
Also, f is continuous and hence it is monotonic. [2 Marks]
In case, f is increasing then for each $x \in \mathbb{R}$,
case(i) $x < f(x)$:
$x < f(x) \implies f(x) < f^2(x) \implies f^2(x) < f^3(x) = x$, which is a contradiction.
case (ii) $x > f(x)$:
$x > f(x) \implies f(x) > f^2(x) \implies f^2(x) > f^3(x) = x$, which is a contradiction.
Hence, if f is increasing then $f(x) = x$, for each $x \in \mathbb{R}$ and hence we get,
$f^2(x) = x$, $\forall x \in \mathbb{R}$. [2 Marks]
Now, let f be decreasing.
$x < y \implies f(x) > f(y) \implies f^2(x) < f^2(y)$ and thus f^2 is increasing.
case (i) $x < f^2(x)$:
$x < f^2(x) \implies f^2(x) < f^4(x) = f(x)$ i.e. $f^2(x) < f(x)$ and thus we get,
$f^4(x) < f^3(x) = x$. This gives, $x < f^2(x) < f^4(x) < f^3(x) = x$. This is a contradiction.
case (ii) $x > f^2(x)$:
$x > f^2(x) \implies f^2(x) > f^4(x)$ i.e. $f^2(x) > f(x)$ and thus we get,
$f^4(x) > f^3(x) = x$. This gives, $x > f^2(x) > f^4(x) > f^3(x) = x$. This is a contradiction.
Hence, we must have $f^2(x) = x$, for all $x \in \mathbb{R}$. [2 Marks]

Part III

1. Find [12]

   (a) $\lim_{n\to\infty} \dfrac{\gcd(1,6) + \gcd(2,6) + \cdots + \gcd(n,6)}{1 + 2 + \cdots + n}$

   (b) $\lim_{n\to\infty} \dfrac{lcm(1,6) + lcm(2,6) + \cdots + lcm(n,6)}{1 + 2 + \cdots + n}$

Solution:
(a) We first calculate the numerator.
$\sum_{n=6k+1}^{6k+6} \gcd(n,6) = \gcd(6k+1,6) + \gcd(6k+2,6) + \gcd(6k+3,6) + \gcd(6k+4,6) + \gcd(6k+5,6) + \gcd(6k+6,6) = 1 + 2 + 3 + 4 + 5 + 6 = 15.$ [3 Marks]

Then $\sum_{k=1}^{m}\sum_{n=6k+1}^{6k+6}\gcd(n,6)=15m$. Let $n=6m$. Now

$$\lim_{n\to\infty}\frac{\gcd(1,6)+\gcd(2,6)+\cdots+\gcd(n,6)}{1+2+\cdots+n}=\lim_{m\to\infty}\frac{(15m)(2)}{6m(6m+1)}=0.$$

[3 Marks]

(b) We first calculate the numerator.
$\sum_{n=6k+1}^{6k+6} lcm(n,6) = lcm(6k+1,6)+lcm(6k+2,6)+lcm(6k+3,6)+lcm(6k+4,6)+ lcm(6k+5,6)+lcm(6k+6,6) = 6(6k+1)+6(3k+1)+6(2k+1)+6(3k+2)+6(6k+5)+6(k+1) = 126k+66.$ [3 Marks]

Then $\sum_{k=1}^{m}(126k+66)=66m+126\frac{m(m+1)}{2}=66m+63m(m+1)$.
Let $n=6m$. Now

$$\lim_{n\to\infty}\frac{lcm(1,6)+lcm(2,6)+\cdots+lcm(n,6)}{1+2+\cdots+n}=\lim_{m\to\infty}\frac{2[66m+63m(m+1)]}{6m(6m+1)}=\frac{63}{18}=\frac{7}{2}.$$

[3 Marks]

2. Let a,b,c be real numbers such that $a^2+b^2+c^2=4$. [12]

(a) Find the value of the determinant of a matrix $A=\begin{pmatrix}a+b & b+c & c+a\\ b+c & c+a & a+b\\ c+a & a+b & b+c\end{pmatrix}$.

(b) Find the maximum and minimum value of the above determinant.

Solution:
Let a,b,c be real numbers such that $a^2+b^2+c^2=4$. Consider

$$D=\begin{vmatrix}a+b & b+c & c+a\\ c+a & a+b & b+c\\ b+c & c+a & a+b\end{vmatrix}$$

We add $R_1+(R_2+R_3)$

$$D=\begin{vmatrix}a+b & b+c & c+a\\ c+a & a+b & b+c\\ b+c & c+a & a+b\end{vmatrix} = 2(a+b+c)\begin{vmatrix}1 & 1 & 1\\ c+a & a+b & b+c\\ b+c & c+a & a+b\end{vmatrix}$$

Now, $R_2-(c+a)R_1, R_3-(b+c)R_1$ gives

$$\begin{aligned} D &= 2(a+b+c)\begin{vmatrix}1 & 1 & 1\\ 0 & b-c & b-a\\ 0 & a-b & a-c\end{vmatrix}\\ &= 2(a+b+c)\left((b-c)(a-c)+(a-b)^2\right)\\ &= 2(a+b+c)\left(a^2+b^2+c^2-ab-bc-ac\right)\end{aligned}$$

[5 Marks]

Let $s=a+b+c$. Hence, $s^2=a^2+b^2+c^2+2(ab+bc+ac)$. [3 Marks]

Hence, $D=f(s)=2s\left(4-\frac{s^2-4}{2}\right)=s(12-s^2)=12s-s^3$.

Then $f'(s)=12-3s^2=0$ gives $s=\pm2$. Now $f''(s)=-6s$ is positive at $s=-2$ and negative at $s=2$. Hence the maximum value of D is 16 at $(-2,0,0)$ and the minimum value of D is -16 at $(2,0,0)$. [4 Marks]

3. For each $t \in \mathbb{R}$ let L_t be the line segment joining the points $(0,1)$ and $(t,0)$. Let P_t be the point of intersection of the line segment L_t with the parabola $y = x^2$. Define function $f : \mathbb{R} \to \mathbb{R}$ as $f(t) = y$ coordinate of point P_t. Answer the following with justification. [13]

   (a) Is f a bounded function?

   (b) Is f a continuous function?

   (c) Find $\lim_{t\to\infty} f(t)$.

   (d) Is f differentiable at 0?

   Solution :
   The equation of line L_t is : $y = 1 - \frac{x}{t}$. Thus, $x-$ coordinate of the point of intersection of L_t with the parabola $y = x^2$ satisfy $1 - \frac{x}{t} = x^2$. This gives $x = -\frac{\frac{1}{t} \pm \sqrt{\frac{1}{t^2} + 4}}{2}$

   For $t > 0$, we get $x = \frac{-1 + \sqrt{1 + 4t^2}}{2t} = \frac{2t}{1 + \sqrt{4t^2 + 1}}$

   and thus, $y = x^2 = \frac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}}$.

   Hence, $f(t) = \frac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}}$ for $t \geq 0$. [5 Marks]

   Further, as $g(x) = x^2$ is an even function from $\mathbb{R}$ to $\mathbb{R}$, it can be observed by definition of f that f is an even function. Hence, for $t < 0$, we get,

   $f(t) = f(-t) = \frac{4(-t)^2}{4(-t)^2 + 2 + 2\sqrt{4(-t)^2 + 1}} = \frac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}}$

   Thus, we get, $f(t) = \frac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}}$, for all $t \in \mathbb{R}$. [3 Marks]

   (a) Since, $4t^2 < 4t^2 + 2 + 2\sqrt{4t^2 + 1}$, $\forall t \in \mathbb{R}$, we get, $0 \leq f(t) < 1$, $\forall t \in \mathbb{R}$. Hence, f is bounded. [1 Mark]

   (b) By continuity of polynomials, square-root function and algebra of continuous functions, function f is a continuous function. [1 Mark]

   (c) $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}} = \frac{4}{4} = 1.$ [1 Mark]

   (d) As $f(0) = 0$, Consider $\lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \frac{4t}{4t^2 + 2 + 2\sqrt{4t^2 + 1}} = 0$ and hence, f is differentiable at 0 and $f'(0) = 0$. [2 Marks]

4. The sequence $\{q_n(x)\}$ of polynomials is defined by $q_1(x) = 1 + x, q_2(x) = 1 + 2x$ and for $m \geq 1$ by

$$q_{2m+1}(x) = q_{2m}(x) + (m+1)xq_{2m-1}(x),$$
$$q_{2m+2}(x) = q_{2m+1}(x) + (m+1)xq_{2m}(x).$$

   Let x_n be the largest real solution of $q_n(x) = 0$. Prove that [13]

   (a) the sequence $\{x_n\}$ is increasing.

   (b) $x_{2m+2} > \frac{-1}{m+1}$ for $m \geq 1$.

   (c) the sequence $\{x_n\}$ converges to 0.

   Solution :
   (a) Since each $q_n(x)$ has non-negative coefficients, any real zero cannot be positive. [1 Mark]

It is easy to check that $q_1(x)$ and $q_2(x)$ each have a negative zero. Suppose for some $n \geq 2$, it has been established that $q_i(0) > 0$ and that each $q_i(x)$ has at least one negative zero and that the greatest such zero x_i satisfies $x_{i-1} < x_i$ $(2 \leq i \leq n)$. Then for $1 \leq i \leq j \leq n,\ q_i(x) > 0$ for $x_j < x \leq 0$. From the recursion relations, it follows that $q_{n+1}(0) > 0$ and $q_{n+1}(x_n) < 0$, so that $q_{n+1}(x)$ has at least one zero in the interval $(x_n, 0)$. Thus, there is a largest real zero x_{n+1} and it satisfies $x_n < x_{n+1} < 0$. [3 Marks]
(b) From the recurrence relations, we find that

$$q_{2m+2}\frac{(-1)}{m+1} = -q_{2m-1}\frac{(-1)}{m+1}.$$

If $q_{2m+2}\frac{(-1)}{m+1} < 0$, then $x_{2m+2} > \frac{(-1)}{m+1}$. On the other hand, if $q_{2m+2}\frac{(-1)}{m+1} > 0$, then $q_{2m-1}\frac{(-1)}{m+1} < 0$, so that $x_{2m+2} > x_{2m-1} > \frac{(-1)}{m+1}$. [5 Marks]
(c) If r is any positive number, we can choose a positive integer m such that $-r < (-1)/(m+1) < 0$. Then for $n \geq 2m+2,\ -r < x_n < 0$. Hence the sequence $\{x_n\}$ converges to 0. [4 Marks]