MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 23/01/2022 Max. Marks: 50
Time: 12.00 noon to 1.30 p.m.
N.B.: Part I carries 20 marks, Part II carries 20 marks and Part III carries 10 marks.

Part I: MCQ with single correct answer

N.B. Each question in Part I carries 2 marks for correct the answer and -1 mark for a wrong answer.

1. Let the sequence $\{x_n\}$ be defined as follows: $x_1 = 1$ and x_n is the smallest prime factor of n. Then the sequence $\{x_n\}$

   (a) is monotonic
   (b) diverges to infinity
   (c) has a convergent subsequence
   (d) is not bounded below

   Ans:(c)

2. The equation $x^6 - x - 1 = 0$ has

   (a) no positive real root
   (b) exactly one positive real root
   (c) exactly two positive real roots
   (d) all roots are real and positive

   Ans:(b)

3. The value of θ $(0 \leq \theta \leq \pi/2)$ for which the number $\dfrac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary is

   (a) $\pi/6$
   (b) $\pi/3$
   (c) $sin^{-1}(\sqrt{3}/4)$
   (d) $sin^{-1}(1/\sqrt{3})$

   Ans:(d)

4. Consider the curve $y = 2x^4 + 7x^3 + 3x - 5$. Let $P_i = (x_i, y_i)$ be four distinct points of intersection of a line with the given curve. Then the value of $\dfrac{x_1 + x_2 + x_3 + x_4}{4}$ is

   (a) $-7/8$
   (b) $-7/2$
   (c) $7/8$
   (d) $7/2$

   Ans:(a)

5. If one root of the equation $x^2 + px + 12 = 0$ is 4 while the equation $x^2 + px + q = 0$ has equal roots, the value of q is

   (a) 4/49
   (b) 49/4
   (c) $-49/4$
   (d) $-4/49$

   Ans:(b)

6. Which of the following equations has greatest number of real solutions?

   (a) $x^3 = 10 - x$
   (b) $x^2 + 5x - 7 = x + 8$
   (c) $7x + 5 = 1 - 3x$
   (d) $e^x = x$

   Ans:(b)

7. Let $\gcd(a, b) = 1$, then $\gcd(a + b, a^2 - ab + b^2) =$

   (a) 2
   (b) 1 or 2
   (c) 1 or 3
   (d) 2 or 3

   Ans:(c)

8. Suppose f is continuous in $[0, 2]$ and differentiable in $(0, 2)$. If $f(0) = 0$ and $|f'(x)| \leq 1/2$ for all $x \in [0, 2]$, then

   (a) $|f(x)| \leq 1$
   (b) $|f(x)| \leq 1/2$
   (c) $f(x) = 2x$
   (d) $f(x) = 3$ for at least one $x \in [0, 2]$.

   Ans:(a)

9. Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. The number of functions from A to B such that exactly one element in B has two pre-images is

   (a) 12
   (b) 18
   (c) 24
   (d) 36

   Ans:(d)

10. Consider a square matrix $A = [a_{ij}]$ of order 3, all whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0. Also $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq 3$. Then the number of such matrices is

    (a) 12
    (b) 9
    (c) 3
    (d) 1

    Ans:(a)

Part II: Numerical Questions

N.B. The answer to each question in Part II is an integer. Each question in Part II carries 2 marks. No marks will be deducted for wrong answer.

1. If the matrix $A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$ satisfies the equation $A^2 - kA + 2I = 0$, then the value of k is .....

   Ans: 1

2. The remainder when $\sum_{r=1}^{100} r!$ is divided by 12 is .....

   Ans: 9

3. $3^1 \times 3^{1/2} \times 3^{1/4} \times 3^{1/8} \times \cdots =$

   Ans: 9

4. Let f be a differentiable real valued function on $(-1, 4)$ such that $f(3) = 5$ and $f'(x) \geq -1$ for all x. Then the greatest possible value of $f(0)$ is .....

   Ans: 8

5. Suppose the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$ is $ax^2 + bx + c$. Then the value of b is .....

   Ans: 5

6. If the sum of the series $a + ar + ar^2 + \cdots$ is 4 and the sum of the series $a^3 + a^3r^3 + a^3r^6 + \cdots$ is 192. Then the value of a is .....

   Ans: 6

7. The sum of the series $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+n} + \cdots$ is .....

   Ans: 2

8. The number of ways to write $5 = a_1 + a_2 + \cdots + a_k$, where all a_i are integers satisfying $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$ is .....

   Ans: 5

9. The number of solutions of $\sin^5 x + \cos^5 x = 1$ in $[0, \pi]$ is .....

   Ans: 2

10. If $A = [a_{ij}]$ is a square matrix of order 5 such that the entry $a_{ij} = 1$ if and only if $i = j$ or $i + j = 6$, and 0 otherwise, then the rank of A is .....

   Ans: 3

Part III: Multiple Select Questions

N.B. Each question in Part III carries 2 marks. No marks will be deducted for wrong answer. Each question may have more than one correct alternatives. A candidate gets 2 marks if he/she selects all the correct answers only and no wrong answers.

1. If the equation $ax^2 + bx + c = 0, (a > 0)$ has two roots α, β such that $\alpha < -2$ and $\beta > 2$, then

   (a) $b^2 - 4ac > 0$

   (b) $c < 0$

   (c) $a + |b| + c < 0$

   (d) $4a + 2|b| + c < 0$

   Ans: (a),(b),(c),(d).

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $a \in \mathbb{R}$. Define $g : [a, \infty) \to \mathbb{R}$ as $g(x) = \sup\{f(t) : t \in [a, x]\}$. Then

   (a) g is continuous

   (b) g is monotonically decreasing

   (c) g is monotonically increasing

   (d) g is differentiable whenever f is differentiable.

   Ans: (a),(c).

3. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non-zero real numbers. We say that $\{a_n\}$ and $\{b_n\}$ are almost equal if $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$. Which of the following sequences are almost equal?

   (a) $a_n = n + \sqrt{n}, b_n = n.$

   (b) $a_n = n^2 + \sqrt{n}, b_n = n.$

   (c) $a_n = n!, b_n = n^n.$

   (d) $a_n = (1 + \frac{1}{n})^n, b_n = e.$

   Ans: (a),(d).

4. For which of the following functions $f : \mathbb{R} \to \mathbb{R}$ the ratio $\dfrac{f(k) - f(m)}{k - m}$ is constant for all $k, m \quad (k \neq m)$?

   (a) $f(x) = x^2 + x$
   (b) $f(x) = x + |x|$
   (c) $f(x) = 4x + 7$
   (d) $f(x) = |x|$

   Ans: (c).

5. Let α be a 2022^{nd} root of unity. Then which of the following are possible values of $1 + \alpha + \alpha^2 + \cdots + \alpha^{2021}$?

   (a) 0
   (b) i
   (c) 2021
   (d) 2022

   Ans:(a),(d)
