MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 07/02/2021 Max. Marks: 50
Time: 12.00 noon to 1.30 p.m.
N.B.: Part I carries 20 marks, Part II carries 20 marks and Part III carries 10 marks.

Part I: MCQ with single correct answer

N.B. Each question in Part I carries 2 marks for correct answer and -1 mark for wrong answer.

1. For matrices X, Y define $[X, Y] = XY - YX$. For $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, rank $[X, Y]$
   (a) equals rank$X$
   (b) is strictly less that rank (XY)
   (c) is strictly greater than rank (XY)
   (d) equals 0.

   Ans:(c)

2. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has
   (a) at least one root in $(0, 1)$.
   (b) one root in $(1, 2)$ and other in $(-1, 0)$.
   (c) both imaginary roots.
   (d) a repeated root

   Ans:(a)

3. Let $f(x)$ be the function defined on $\mathbb{R}$ such that $f(x) = x$, for all $x \leq 1$ and $f(x) = ax^2 + bx + c$, for $x > 1$. The triples (a, b, c) such that $f(x)$ is differentiable at all real x are of the form:
   (a) $(a, 1 - 2a, a)$
   (b) $(1, 0, 0)$ only
   (c) $(a, -2a, a)$
   (d) $(0, 1, -1)$ only

   Ans:(a)

4. If $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, for all $x \in (0,\ 1)$, then $\int_0^x f(t)dt$ equals
   (a) $\tan^{-1}(x)$
   (b) $\sin^{-1}(x)$
   (c) $\sin(x)$
   (d) $\tan(x)$

Ans:(a)

5. The system of equations

$$2x + py + 6z = 8$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

has no solution for

(a) $p \neq 2, q \neq 3$
(b) $p \neq 2, q = 3$
(c) $p = 2, q = 3$
(d) $p = 2, q \neq 3$

Ans:(b)

6. Let f be any differentiable function on $\mathbb{R}$ with $f(-2) = 16, f(4) = 4$ and $f(8) = 24$. For which of the following three values of λ the equation $f'(x) = \lambda$ must have a solution?

(a) $0, 5, 10$
(b) $-5, 0.8, 3$
(c) $-2, 0.8, 5$
(d) $-2, 0, 14$

Ans:(c)

7. If $\int \frac{f(x)}{\log \sin x} dx = \log \log \sin x$, then $f(x) =$

(a) $\sin x$
(b) $\cos x$
(c) $\log \sin x$
(d) $\cot x$

Ans:(d)

8. Let $x_n = \{1, -1, 2, -2, 3, -3, \cdots\}$ and $y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}, n \geq 1$, then the sequence $\{y_n\}$ is

(a) monotonic but not convergent.
(b) convergent.
(c) bounded but not convergent.
(d) not bounded.

Ans:(c)

9. Let $A = \{(x, y) \in \mathbb{R}^2 \ : \ xy \in \mathbb{Z}\}$ and $B = \{(x, y) \in \mathbb{R}^2 \ : \ x + y \in \mathbb{Q}\}$. Then

(a) $A \cap B$ is countable and B is uncountable.
(b) $A \cap B$ is uncountable.
(c) A is countable.
(d) B is countable.

Ans: (a)

10. We have two glasses. One having milk and other having water in it, in exactly the same quantities. One spoon of milk is transferred to water and then one spoon of mixture is replaced in milk. Then which of the following is true?

 (a) The amount of milk in water is bigger than the amount of water in milk.
 (b) The amount of milk in water is smaller than the amount of water in milk.
 (c) The amount of milk in water is equal to the amount of water in milk.
 (d) The information is insufficient.

 Ans:(c)

Part II: Numerical Questions

N.B. The answer to each question in Part II is an integer. Each question in Part II carries 2 marks. No marks will be deducted for wrong answer.

1. For a complex number z, if $|z+4| \leq 3$, then the maximum value of $|z+1|$ is .....

 Ans: 6

2. The number of continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $(f(x))^2 = xf(x)$ is .....

 Ans: 4

3. For any positive integer n, let $S(n)$ be the sum of digits of n and $T(n) = S(n) + n$. The smallest value of n such that $T(n) = T(n+k)$ for some positive integer k is .....

 Ans: 91

4. $\lim_{x \to 0} \dfrac{\cos(4x) - 1}{x^2} = ...$

 Ans: -8

5. If $f(x) = x^3 + \sin x$, then the value of $\int_{-\pi/2}^{\pi/2} (x^2+1)(f(x) + f''(x))dx$ is .....

 Ans: 0

6. The **sum** of all real solutions of the equation $(x^2 - 5x + 5)^{(x^2+4x-60)} = 1$ is .....

 Ans: 3

7. Let $a_1 = \dfrac{10}{11}$ and $a_n = \dfrac{10}{11} + \dfrac{9}{11}a_{n-1}$ for $n \geq 2$, then $\lim_{n \to \infty} a_n = ...$

 Ans: 5

8. Three of the roots of a polynomial $p(X)$ with rational coefficients are: 2020, $\sqrt{7}$, $2021 - 2022i$. The least possible degree of $p(X)$ is .....

 Ans: 5

9. The greatest negative integer satisfying $x^2 - 4x - 77 < 0$ and $x^2 > 4$ is .....

 Ans: -3

10. The number of real values of a such that there exists a real number x satisfying the equation $a^2 - 2a \sin x + 1 = 0$ is

 Ans: 2

Part III: Multiple Select Questions

N.B. Each question in Part III carries 2 marks. No marks will be deducted for wrong answer. Each question may have more than one correct alternatives. A candidate gets 2 marks if he/she selects all the correct answers only and no wrong answers.

1. A four letter word is converted into a matrix form by writing its letters, say $ABCD$ as the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Each such matrix is then replaced by the corresponding letter in the alphabet via the rule $A \mapsto 1$, $B \mapsto 2, \ldots, Z \mapsto 26$. The resulting matrix is further reduced modulo 5. If the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ has been obtained by the above procedure, the following words are among the possible original words

   (a) GOOD
   (b) QEOS
   (c) GEOD
   (d) QEED

   Ans: (a),(b),(c),(d).

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all real numbers x, y. Then

   (a) f^{-1} may not exist
   (b) f is one-one
   (c) f is onto
   (d) f^{-1} is continuous

   Ans: (b),(c),(d).

3. Let $p(x) = x^2 + x + 2020$. If $x_n = p(n), y_n = x_{n+1} - x_n$ and $z_n = y_{n+1} - y_n$, then

   (a) The sequence $\{y_n\}$ is convergent.
   (b) The sequence $\{z_n\}$ converges to zero.
   (c) The sequence $\{z_n\}$ is constant.
   (d) The sequence $\{y_n\}$ is monotonic.

   Ans: (c),(d).

4. Let $x = 0.a_1a_2a_3a_4\cdots$ be the decimal expansion of $x \in (0, 1)$, then x is rational if

   (a) $a_n = 0$ if $n = k^2$ and $a_n = 1$ if $n \neq k^2$ for any positive integer k.
   (b) $a_n = 0$ if n is odd and $a_n = 1$ if n is even.
   (c) $a_n = 0$ if $n = k!$ and $a_n = 1$ if $n \neq k!$ for any positive integer k.
   (d) $a_n = 0$ if $n \leq 2020$ and $a_n = 1$ if $n \geq 2021$.

   Ans: (b),(d).

5. If $1, w_1, w_2, w_3, w_4, w_5$ are distinct roots of $x^6 - 1$, then

   (a) $1 + w_i + w_i^2 + w_i^3 + w_i^4 + w_i^5 = 0$ for $i = 1, 2, 3, 4, 5$.
   (b) $1 + w_i^2 + w_i^4 + w_i^6 + w_i^8 + w_i^{10} = 0$ for $i = 1, 2, 3, 4, 5$.
   (c) $1 + w_i^3 + w_i^6 + w_i^9 + w_i^{12} + w_i^{15} = 0$ for $i = 1, 2, 3, 4, 5$.
   (d) $1 + w_i^5 + w_i^{10} + w_i^{15} + w_i^{20} + w_i^{25} = 0$ for $i = 1, 2, 3, 4, 5$.

   Ans: (a),(d).