MADHAVA MATHEMATICS COMPETITION, 8 January 2012

Part I

N.B. Each question in Part I carries 2 marks.

1. Let n be a fixed positive integer. The value of k for which $\int_1^k x^{n-1}dx = \frac{1}{n}$ is
   (a) 0 (b) 2^n (c) $(\frac{2}{n})^{\frac{1}{n}}$ (d) $2^{\frac{1}{n}}$

2. Let $S = \{a, b, c\}$, $T = \{1, 2\}$. If m denotes the number of one-one functions and n denotes the number of onto functions from S to T, then the values of m and n respectively are
   (a) 6,0 (b) 0,6 (c) 5,6 (d) 0,8.

3. In the binary system, $\frac{1}{2}$ can be written as
   (a) $0.01111\cdots$ (b) $0.01000\cdots$ (c) $0.0101\cdots$ (d) None of these

4. For the equation $|x|^2 + |x| - 6 = 0$
   (a) there is only one root. (b) the sum of the roots is -1.
   (c) the sum of the roots is 0. (d) the product of the roots is -6.

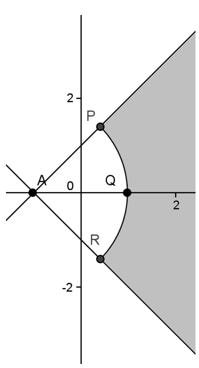
5. The value of $\lim_{n\to\infty} \frac{1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!}{(n+1)!}$
   (a) 1 (b) 2 (c) $\frac{1}{2}$ (d) does not exist.

6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f'(x) > g'(x)$ for every x. Then the graphs $y = f(x)$ and $y = g(x)$
   (a) intersect exactly once. (b) intersect at most once.
   (c) do not intersect. (d) could intersect more than once.

7. The function $|x|^3$ is
   (a) differentiable twice but not thrice at 0. (b) not differentiable at 0.
   (c) three times differentiable at 0. (d) differentiable only once at 0.

8. Let A be the $n \times n$ matrix $(n \geq 2)$, whose $(i, j)^{th}$ entry is $i + j$ for all $i, j = 1, 2, \cdots, n$. The rank of A is
   (a) 2 (b) 1 (c) n (d) $n - 1$.

9. Let $A = -1 + 0i$ be the point in the complex plane. Let PQR be an arc with centre at A and radius 2. If $P = -1 + \sqrt{2} + \sqrt{2}i, Q = 1 + 0i$ and $R = -1 + \sqrt{2} - \sqrt{2}i$



   then the shaded region is given by
   (a) $|z + 1| < 2, |\arg(z + 1)| < \frac{\pi}{2}$
   (b) $|z - 1| < 2, |\arg(z - 1)| < \frac{\pi}{2}$
   (c) $|z + 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$
   (d) $|z - 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$

10. The solution of $\frac{dy}{dx} = a^{x+y}$ is
    (a) $a^x - a^{-y} = c$ (b) $a^{-x} + a^{-y} = c$ (c) $a^{-x} - a^y = c$ (d) $a^x + a^{-y} = c$

Part II

N.B. Each question in Part II carries 5 marks.

1. Let $f : [0,4] \to [3,9]$ be a continuous function. Show that there exists x_0 such that $f(x_0) = \frac{3x_0 + 6}{2}$.

2. Suppose a, b, c are all real numbers such that $a+b+c > 0$, $abc > 0$ and $ab+bc+ac > 0$. Show that a, b, c are all positive.

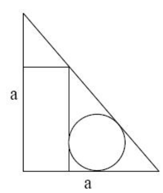
3. Let f be a continuous function on $[0,2]$ and twice differentiable on $(0,2)$. If $f(0) = 0, f(1) = 1$ and $f(2) = 2$, then show that there exists x_0 such that $f''(x_0) = 0$.

4. Integers $1, 2, \cdots, n$ are placed in such a way that each value is either bigger than all preceding values or smaller than all preceding values. In how many ways this can be done? (For example in case of $n = 5$, 3 2 4 1 5 is valid and 3 2 5 1 4 is not valid.)

Part III

N.B. Each question in Part III carries 12 marks.

1. Consider an isosceles right triangle with legs of fixed length a.



Inscribe a rectangle and a circle inside the triangle as indicated in the figure. Find the dimensions of the rectangle and the radius of the circle which make the total area of the rectangle and circle maximum.

2. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, one-one function. If there exists a positive integer n such that $f^n(x) = x$, for every $x \in \mathbb{R}$, then prove that either $f(x) = x$ or $f^2(x) = x$. (Note that $f^n(x) = f(f^{n-1}(x))$.)

3. Consider $f(x) = \frac{x^3}{6} + \frac{x^2}{2} + \frac{x}{3} + 1$. Prove that $f(x)$ is an integer whenever x is an integer. Determine with justification, conditions on real numbers a, b, c and d so that $ax^3 + bx^2 + cx + d$ is an integer for all integers x.

4. Suppose $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & b & d \\ 1 & b & d+1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $U = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$. Find conditions on A and U such that the system $AX = U$ has no solution.

5. Let A and B be finite subsets of the set of integers. Show that $|A+B| \geq |A| + |B| - 1$. When does equality hold? (Here $A + B = \{x + y : x \in A,\ y \in B\}$. Also, $|S|$ denotes the number of elements in the set S.)