

MADHAVA MATHEMATICS COMPETITION (Second Round)
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 03/03/2022

Max. Marks: 50

Time: 12.00 noon to 2.00 p.m.

N.B.: Part I carries 30 marks and Part II carries 20 marks.

Part I

N.B. Each question in Part I carries 6 marks.

1. Let the positive integers a, b, c be such that $a \geq b \geq c$ and $(a^x - b^x - c^x)(x - 2) > 0$ for all $x \neq 2$. Show that a, b, c are sides of a right angled triangle.
2. Find all real numbers x, y such that the fractional part of $\frac{x + 4y + 1}{x^2 + y^2 + 19}$ is $\frac{1}{2}$.
3. Let f be a quadratic polynomial. Show that there exist quadratic polynomials g, h such that $f(x)f(x + 1) = g(h(x))$.
4. Determine the number of all $m \times n$ matrices with entries 0 or 1 such that the number of 1's in each row and the number of 1's in each column are all even.
5. Find all non-negative integer solutions to the system of equations

$$3x^2 - 2y^2 - 4z^2 + 54 = 0$$

$$5x^2 - 3y^2 - 7z^2 + 74 = 0$$

Part II

N.B. Each question in Part II carries 10 marks.

1. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a differentiable function such that f' is continuous and $f(0) = 0, f(1) = 1$.

(a) Show that there exists x_1 in $(0, 1)$ such that $\frac{1}{f'(x_1)} = 1$. [1]

(b) Show that there exist distinct x_1, x_2 in $(0, 1)$ such that $\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2$. [4]

(c) Show that for a positive integer n , there exist distinct x_1, x_2, \dots, x_n in $(0, 1)$ such that $\sum_{i=1}^n \frac{1}{f'(x_i)} = n$. [5]

2. Let \mathcal{P}_n denote the collection of polynomials of degree n such that the polynomial and all its derivatives have integer roots.

(a) Find a polynomial in \mathcal{P}_2 having at least two distinct roots. [2]

(b) Find a polynomial in \mathcal{P}_3 having at least two distinct roots. [3]

(c) For any polynomial P in \mathcal{P}_n , show that the arithmetic mean of all roots of P is also an integer. [5]
