MADHAVA MATHEMATICS COMPETITION (Second Round)
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 15/02/2021 Max. Marks: 50
Time: 12.00 noon to 2.00 p.m.
N.B.: Part I carries 30 marks and Part II carries 20 marks.

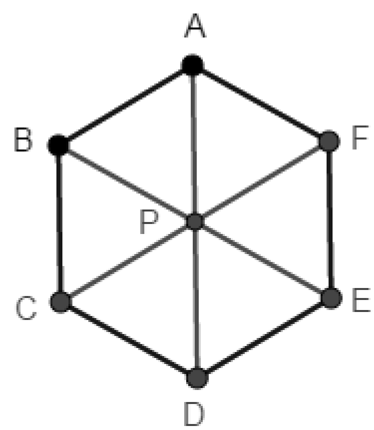
Part I

N.B. Each question in Part I carries 6 marks.

1. Let $p(x) = x^4 - ax^3 + bx^2 - cx + 480$ be a polynomial whose all zeros are integers greater than 1. Let further, $|p(4)| + |p'(4)| = 0$. Find all such polynomials. Also find the least and greatest possible value of a.

2. Let $M = \begin{pmatrix} 1/4 & 1/8 & 1/16 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$. Find the $\lim\limits_{n\to\infty} M^n \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

3. In the diagram given below, the points A, B, C, D, E, F and P represent cities and edges joining them represent roads connecting the cities.



Two cities are said to be adjacent if there is a single edge joining them. (For example, the cities B, C are adjacent, but B, E are not). A tourist moves to an adjacent city on each day. Starting with P, the tour ends if the tourist happens to come back at P.

(a) Find the probability that the tour ends on the second day.

(b) Find the probability that the tour ends on the third day.

(c) For any natural number n, find the probability that the tour ends on n^{th} day.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that $f(c) = 0$ for some $c > 0$. Let $b > 0$ and $x_0 > c$.

   (a) Consider a line L_0 joining $(0, b)$ and $(x_0, 0)$. Prove that there exists a unique real number x_1 such that $(x_1, f(x_1))$ lies on L_0.

   (b) Let L_1 be a line joining $(0, b)$ and $(x_1, 0)$. Define the sequence $\{x_n\}$ inductively applying the above procedure. Prove that the sequence $\{x_n\}$ is convergent. Further, show that $\{x_n\}$ converges to c.

5. Prove that every integer between 1 and $n!$, $(n \geq 3)$ can be expressed as the sum of at most n distinct divisors of $n!$.

Part II

N.B. Each question in Part II carries 10 marks.

1. (a) Show that there does not exist a function $f : (0, \infty) \to (0, \infty)$ such that $f''(x) \leq 0$ for all x and $f'(x_0) < 0$ for some x_0.

   (b) Let $k \geq 2$ be any integer. Show that there does not exist an infinitely differentiable function $f : (0, \infty) \to (0, \infty)$ such that $f^{(k)}(x) \leq 0$ for all x and $f^{(k-1)}(x_0) < 0$ for some x_0. Here, $f^{(k)}$ denotes the k^{th} derivative of f.

2. Prove that every monic polynomial $f(x)$ of degree n over $\mathbb{R}$ can be expressed as an arithmetic mean of two monic polynomials of degree n over $\mathbb{R}$ each having n real roots.
