MADHAVA MATHEMATICS COMPETITION 2021 (Second Round)
Solutions and scheme of marking

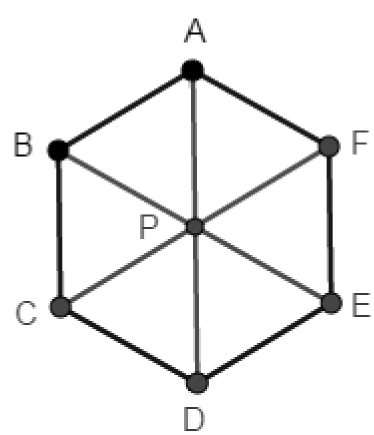
Part I

N.B. Each question in Part I carries 6 marks.

1. Let $p(x) = x^4 - ax^3 + bx^2 - cx + 480$ be a polynomial whose all zeros are integers greater than 1. Let further, $|p(4)| + |p'(4)| = 0$. Find all such polynomials. Also find the least and greatest possible value of a.
   Solution: The condition $|p(4)| + |p'(4)| = 0$ implies that $p(4) = 0$ and $p'(4) = 0$. Therefore 4 is a repeated root of $p(x)$. [2]
   Suppose the roots of the polynomial $p(x)$ are $4, 4, \alpha, \beta$. Thus, $\alpha\beta = 30$. Now α, β being integers greater than 1, the only possible values of α, β are $3, 10$ or $5, 6$ or $2, 15$. [2]
   It is now clear that the least possible value of a is $5 + 6 + 8 = 19$ and the greatest possible value of a is $15 + 2 + 8 = 25$. [1]
   There are three polynomials satisfying the given conditions:
   $p_1(x) = (x-4)^2(x-3)(x-10), p_2(x) = (x-4)^2(x-5)(x-6),$
   $p_3(x) = (x-4)^2(x-2)(x-15).$ [1]

2. Let $M = \begin{pmatrix} 1/4 & 1/8 & 1/16 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$. Find the $\lim_{n\to\infty} M^n \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$
   Solution: Observe that $X = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ [2]
   Further, it can be seen by induction that $M^n \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ and
   $M^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4^n \\ 0 \\ 0 \end{pmatrix}$ for all $n \in \mathbb{N}$. [3]
   Thus, $\lim_{n\to\infty} M^n \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \lim_{n\to\infty} M^n \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \lim_{n\to\infty} M^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$. [1]

3. In the diagram given below, the points A, B, C, D, E, F and P represent cities and edges joining them represent roads connecting the cities.



Two cities are said to be adjacent if there is a single edge joining them. (For example, the cities B, C are adjacent, but B, E are not). A tourist moves to an adjacent city on each day. Starting with P, the tour ends if the tourist happens to come back at P.

(a) Find the probability that the tour ends on the second day.

(b) Find the probability that the tour ends on the third day.

(c) For any natural number n, find the probability that the tour ends on n^{th} day.

Solution: Let P_n denote the probability that the tour ends on the n^{th} day. Clearly, the value of P_1 is 0. (a) It is clear that $P_2 = 1/3$, as no matter where the tourist goes on day 1, from that vertex (any of A, B, C, D, E, F) out of three options, there is exactly one option to come back to P. [1]
(b) We first note that in this case, tourist does not reach P on day 1 or day 2. By symmetry, without loss we may assume that on day 1, the tourist reaches one of the cities A, B, C, D, E, F with probability 1.
Suppose the tourist is at A at the end of day 1. Out of three options at A, the tourist has two options B or F in order not to reach P on day 2. This event has probability 2/3. Again by symmetry, we may assume that the tourist is at B at the end of day 2. Out of three options at B, there is exactly one option to come back to P. This event has probability 1/3. By multiplication theorem, required probability is $P_3 = 1 \times 2/3 \times 1/3 = 2/9$. [3]
(c) The argument for the general case is similar to that in part (b). Indeed, on day 2, day 3, $\cdots$, day $(n-1)$ the tourist has two options out of three in order not to reach P. Each of these events has probability 2/3 and again as in part (b) on n^{th} day the event has probability 1/3.
Therefore $P_n = 1 \times 2/3 \times 2/3 \times \cdots \times 2/3 \times 1/3 = 2^{n-2}/3^{n-1}$. [2]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that $f(c) = 0$ for some $c > 0$. Let $b > 0$ and $x_0 > c$.

(a) Consider a line L_0 joining $(0, b)$ and $(x_0, 0)$. Prove that there exists a unique real number x_1 such that $(x_1, f(x_1))$ lies on L_0.

(b) Let L_1 be a line joining $(0, b)$ and $(x_1, 0)$. Define the sequence $\{x_n\}$ inductively applying the above procedure. Prove that the sequence $\{x_n\}$ is convergent. Further, show that $\{x_n\}$ converges to c.

Solution: Define a function $g_0 : \mathbb{R} \to \mathbb{R}$ by $g_0(x) = x_0 - \frac{x_0}{b} f(x) - x$. As f is continuous so is g_0. Further, $g_0(c) = x_0 - c > 0$ and $g_0(x_0) = -\frac{x_0}{b} f(x_0) < 0$.
By Intermediate Value Theorem, there exist $x_1 \in (c, x_0)$ such that $g_0(x_1) = 0$ and hence $x_1 = x_0 - \frac{x_0}{b} f(x_1)$. As f is strictly increasing function, we have unique such x_1. Since the equation of the line L_0 is $y = b - \frac{b}{x_0} x$, the point $(x_1, f(x_1))$ lies on the line L_0.
Now define the function $g_1 : \mathbb{R} \to \mathbb{R}$ by $g_1(x) = x_1 - \frac{x_1}{b} f(x) - x$. Using the similar argument as above, we get the unique point $(x_2, f(x_2))$ which lies on the line L_1.
Applying the above procedure repeatedly, we get a sequence (x_n) such that

$x_{n+1} = x_n - \dfrac{x_n}{b} f(x_{n+1})$ and $x_{n+1} < x_n$, $\forall n \in \mathbb{N}$. [3]
Thus the sequence (x_n) is decreasing and bounded below by c and hence it is a convergent sequence.
Let $p = \lim_{n\to\infty} x_n$. Then, using the relation, $x_{n+1} = x_n - \dfrac{x_n}{b} f(x_{n+1})$, we get, $f(p) = 0$.
But, f is strictly increasing and $f(c) = 0$, hence we must have $c = p$. Thus, (x_n) converges to c. [3]

5. Prove that every integer between 1 and $n!$, $(n \geq 3)$ can be expressed as the sum of at most n distinct divisors of $n!$.
Solution: It can be seen that the result is true for $n = 3$.
Assume that the result is true for $n - 1$ for $n \geq 4$. Consider any k, $1 < k < n!$.
Let $k = nq + r$ where $0 \leq r < n$. [3]
Observe that $0 < q \leq \dfrac{k}{n} < \dfrac{n!}{n} = (n-1)!$.
By induction hypothesis, there exists $d_1, d_2, \cdots, d_m$ such that $q = d_1 + d_2 + \cdots + d_m$ where each d_i divides $(n-1)!$ and $m \leq n - 1$.
We may assume $1 \leq d_1 < d_2 < \cdots < d_m$.
If $r = 0$, then $k = nq = nd_1 + nd_2 + \cdots + nd_m$ where each nd_i divides $n!$, nd_i are all distinct and $m \leq n - 1 < n$.
If $r \neq 0$, then $1 \leq r \leq n - 1 < n$. This implies r divides $n!$ and $r < n < nd_1$.
Hence $k = nq + r = r + nd_1 + nd_2 + \cdots + nd_m$, where $m + 1 \leq n$, as required.
Thus the result is true for n and hence for every natural number. [3]

Part II

N.B. Each question in Part II carries 10 marks.

1. (a) Show that there does not exist a function $f : (0, \infty) \to (0, \infty)$ such that $f''(x) \leq 0$ for all x and $f'(x_0) < 0$ for some x_0.

   (b) Let $k \geq 2$ be any integer. Show that there does not exist an infinitely differentiable function $f : (0, \infty) \to (0, \infty)$ such that $f^{(k)}(x) \leq 0$ for all x and $f^{(k-1)}(x_0) < 0$ for some x_0. Here, $f^{(k)}$ denotes the k^{th} derivative of f.

First Solution:
(a) Suppose there exists a function $f : (0, \infty) \to (0, \infty)$ such that $f''(x) \leq 0$ for all x and $f'(x_0) < 0$ for some x_0. Since f' is a decreasing function and $f'(x_0) < 0$, we get $f'(x) < 0$ for all $x > x_0$.
By Fundamental Theorem of Calculus, we have $f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt$.
Thus there exists $x_1 > x_0$ such that $f(x_1) < 0$, a contradiction. [6]
(b) Suppose there exists an infinitely differentiable function $f : (0, \infty) \to (0, \infty)$ such that $f^{(k)}(x) \leq 0$ for all x and $f^{(k-1)}(x_0) < 0$ for some x_0. Since $f^{(k-1)}$ is a decreasing function and $f^{(k-1)}(x_0) < 0$, we get $f^{(k-1)}(x) < 0$ for all $x > x_0$.
By Fundamental Theorem of Calculus, we have $f^{(k-2)}(x) = f^{(k-2)}(x_0) + \int_{x_0}^{x} f^{(k-1)}(t)dt$.
Thus there exists $x_1 > x_0$ such that $f^{(k-2)}(x) < 0$ for $x > x_1$. By a repeated application of the Fundamental Theorem of Calculus, we get that, $f(x) < 0$ for some real number x, a contradiction. [4]

Second Solution: Suppose there exists a function $f : (0, \infty) \to (0, \infty)$ such that $f''(x) \leq 0$ for all x and $f'(x_0) < 0$ for some x_0. Thus f' is a decreasing function. Define $g(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$.

Note that $g(x_0) = 0$ and $g'(x) = f'(x) - f'(x_0) < 0$ for all $x > x_0$.
Since g is a decreasing function on (x_0, ∞), we get $g(x) \leq 0$ for all $x > x_0$.
Now $f'(x_0) < 0$ implies, there exists $x_1 > x_0$ such that $f(x_1) < 0$. [6]
(b) For the general $k \geq 2$, proof can be given by similar argument. [4]

2. Prove that every monic polynomial $f(x)$ of degree n over $\mathbb{R}$ can be expressed as an arithmetic mean of two monic polynomials of degree n over $\mathbb{R}$ each having n real roots.

Solution:

Case 1: If $deg f(x) = 1$ then $f(x) = x + a$ for some $a \in \mathbb{R}$. Take $P(x) = x$ and $Q(x) = x + 2a$. Then, both $P(x)$ and $Q(x)$ are of degree 1, having real root and $f(x) = \dfrac{P(x) + Q(x)}{2}$. [1]

Assume deg $f(x) > 1$.

Let $Q(x) = x^n - k(x-2)(x-4)\ldots(x-2(n-1))$ and $P(x) = 2f(x) - Q(x)$, where $k > 0$ is chosen sufficiently large so that $k > x^n$ and $P(x)$ and $Q(x)$ have opposite signs for all $x \in [0, 2n]$. Note that the choice of $P(x)$ is made in such a way that if $Q(x)$ has n real roots then $P(x)$ also will have n real roots.

Case 2: Let deg $f(x) = 2$

Here, $Q(x) = x^2 - k(x-2)$. Observe that $Q(2) > 0$ and $Q(3) < 0$. Thus, $Q(x)$ has root in the interval $(2, 3)$ and consequently, will have two real roots. [2]

Case 3: Let deg $f(x) = 3$

Here, $Q(x) = x^3 - k(x-2)(x-4)$. Observe that $Q(0) < 0$ and $Q(2) > 0$ and $Q(4) > 0, Q(6) < 0$. Thus $Q(x)$ has a root in each of the intervals $(0, 2), (4, 6)$. Consequently $Q(x)$ has 3 real roots. [2]

Case 4: Suppose n is even and $n \geq 4$.

Observe that $Q(2) > 0$, $Q(4) > 0$, $\ldots$, $Q(2n-2) > 0$ and $Q(1) > 0$, $Q(3) < 0$, $Q(5) > 0$, $Q(7) < 0$, $\ldots$, $Q(2n-3) > 0$, $Q(2n-1) < 0$ and $Q(x) > 0$ for sufficiently large $x > 2n-2$.

Therefore $Q(x)$ has two roots in each of the intervals $(2, 4)$, $(6, 8)$, $\ldots$, $(2n-6, 2n-4)$ and one real root in the interval $(2n-2, 2n)$. Therefore there are two roots in each of $(n-2)/2$ intervals and one root in $(2n-2, 2n)$ resulting into $n-1$ real roots. Consequently $Q(x)$ has n real roots.

Case 5: Suppose n is odd and $n \geq 5$.

Observe that $Q(2) > 0$, $Q(4) > 0$, $\ldots$, $Q(2n-2) > 0$ and $Q(0) < 0$, $Q(5) < 0$, $Q(9) < 0$, $\ldots$, $Q(2n-5) < 0$ and $Q(x) < 0$ for sufficiently large $x > 2n-2$.

Therefore $Q(x)$ has two roots in each of the intervals $(4, 6)$, $(8, 10)$, $\ldots$, $(2n-6, 2n-4)$ and one real root in each of the intervals $(0, 2)$ and $(2n-2, 2n)$. Therefore there are two roots in each of $(n-3)/2$ intervals and one each in $(0, 2)$ and $(2n-2, 2n)$ resulting into $n-1$ real roots. Consequently $Q(x)$ has n real roots. [5]

(Note: 5 marks for both cases and 3 marks for only one case)