MADHAVA MATHEMATICS COMPETITION, 12th January 2020
Solutions and scheme of marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let A be a non-empty subset of real numbers and $f : A \to A$ be a function such that $f(f(x)) = x$ for all $x \in A$. Then $f(x)$ is
   A) a bijection B) one-one but not onto
   C) onto but not one-one D) neither one-one nor onto.
   Answer: A
   If $f(x) = f(y)$, then $f(f(x)) = f(f(y))$ implies $x = y$. Therefore f is one-one function. By definition, f is onto. Hence f is a bijection.

2. If $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x + y) = f(xy)$ for all $x, y \in \mathbb{R}$ and $f(3/4) = 3/4$, then $f(9/16) =$
   A) 9/16 B) 0 C) 3/2 D) 3/4.
   Answer: D
   If we put $y = 0$, then $f(x) = f(0)$ for all $x \in \mathbb{R}$. This implies that f is a constant function.

3. The area enclosed between the curves $y = \sin^2 x$ and $y = \cos^2 x$ in the interval $0 \le x \le \pi/2$ is
   A) 2 B) 1/2 C) 1 D) 3/4.
   Answer: C
   Area $= \displaystyle\int_0^{\pi/4} (\cos^2 x - \sin^2 x)dx + \int_{\pi/4}^{\pi/2} (\sin^2 x - \cos^2 x)dx$
   $= \displaystyle\int_0^{\pi/4} \cos 2x dx - \int_{\pi/4}^{\pi/2} \cos 2x dx = 1/2 + 1/2 = 1.$

4. The number of ordered pairs (m, n) of all integers satisfying $\dfrac{m}{12} = \dfrac{12}{n}$ is
   A) 15 B) 30 C) 12 D) 10.
   Answer: B
   We have $mn = 144$ and $144 = 2^4 3^2$. There are 15 divisors of 144 which are positive integers. Including negative integers total 30 pairs are there.

5. Suppose $2 \log x + \log y = x - y$. Then the equation of the tangent line to the graph of this equation at the point $(1, 1)$ is
   A) $x + 2y = 3$ B) $x - 2y = 3$ C) $2x + y = 3$ D) $2x - y = 3$.
   Answer: A
   Consider $2 \log x + \log y = x - y$. Differentiating this equation with respect to x,

we get $\frac{2}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$. At point $(1,1)$ the slope of tangent is $-1/2$. The equation of tangent is $x + 2y = 3$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sin[x]$, where $[x]$ denotes the greatest integer less than or equal to x. Then
A) f is a 2π-periodic function B) f is a π-periodic function
C) f is a 1-periodic function D) f is not a periodic function.
Answer: D
If f is a periodic function with period T, then $\sin[x+T] = \sin[x]$. This implies $[x+T] - [x] = 2n\pi$ for some integer n, which is not possible.

7. For how many integers a with $1 \le a \le 100$, a^a is a square?
A) 50 B) 51 C) 55 D) 56.
Answer: C
Case 1: all even integers. There are 50 even integers between 1 and 100.
Case 2: a is an odd number which is a square i. e. 1,9,25,49,81.
Therefore total number is 55.

8. $\lim_{x\to 0} x\left[\frac{1}{x}\right]$
A) 0 B) 1 C) -1 D) does not exist.
Answer: B
$\lim_{x\to 0}\left(x\left[\frac{1}{x}\right] - 1\right) = \lim_{x\to 0}\left(x\left[\frac{1}{x}\right] - x\frac{1}{x}\right) = \lim_{x\to 0} x\left(\left[\frac{1}{x}\right] - \frac{1}{x}\right) = 0.$

9. If α and β are the roots of $x^2 + 3x + 1$ then $\left(\frac{\alpha}{\beta+1}\right)^2 + \left(\frac{\beta}{\alpha+1}\right)^2$ equals
A) 19 B) 18 C) 20 D) 17.
Answer: B
Observe that $\alpha + \beta = -3, \alpha\beta = 1, \alpha^2 + 3\alpha + 1 = 0, \beta^2 + 3\beta + 1 = 0$. Therefore $(\alpha+1)^2 = \alpha^2 + 2\alpha + 1 = -\alpha, (\beta+1)^2 = \beta^2 + 2\beta + 1 = -\beta$.
$\left(\frac{\alpha}{\beta+1}\right)^2 + \left(\frac{\beta}{\alpha+1}\right)^2 = \left(\frac{\alpha^2}{-\beta}\right) + \left(\frac{\beta^2}{-\alpha}\right) = \frac{1+3\alpha}{\beta} + \frac{1+3\beta}{\alpha}$
$= \frac{(1+3\alpha)\alpha + (1+3\beta)\beta}{\alpha\beta} = 3(\alpha^2+\beta^2) + (\alpha+\beta) = 18.$

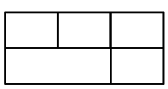
10. The equation $z^3 + iz - 1 = 0$ has
A) no real root B) exactly one real root
C) three real roots D) exactly two real roots.
Answer: A
If x is a real root, then $x^3 + ix - 1 = 0$. This implies that $x = 0$ and $x^3 = 1$, which is not possible.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $a_1, a_2, \cdots$ be a sequence of natural numbers. Let (a, b) denote the greatest common divisor (gcd) of a and b. If $(a_m, a_n) = (m, n)$ for all $m \neq n$, prove that $a_n = n$ for all $n \in \mathbb{N}$.
Solution: Note that $(a_n, a_{n^2}) = (n, n^2) = n$. Therefore n divides a_n. [2]
Let $a_n = mn$. Then $(a_n, a_{mn}) = (n, mn) = n$ and $(a_m, a_{mn}) = (m, mn) = m$. [2]
This implies that mn divides a_{mn}. Since mn divides both a_{mn} and a_n, it divides their gcd n. Hence $m = 1$ and thus $a_n = n$. [2]

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(z)f(iz) = z^2$ for all $z \in \mathbb{C}$. Prove that $f(z) + f(-z) = 0$ for all $z \in \mathbb{C}$. Find such a function.
Solution: It is given that $f(z)f(iz) = z^2$ for all $z \in \mathbb{C}$. Replacing z by iz, we get $f(iz)f(-z) = -z^2$. Adding these two expressions we get, $f(iz)[f(z)+f(-z)] = 0$.
From $f(z)f(iz) = z^2$ we deduce that $f(z) = 0$ if and only if $z = 0$.
If $z \neq 0$, then $f(iz) \neq 0$ and so $f(z) + f(-z) = 0$.
If $z = 0$, then $f(z) + f(-z) = 2f(0) = 0$. Thus $f(z) + f(-z) = 0, \ \forall z \in \mathbb{C}$. [4]
Example: $f(z) = \left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) z$ or $f(z) = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) z$ [2]

3. Let n be a positive integer. Line segments can be drawn parallel to edges of a given rectangle. What is the minimum number of line segments (not necessarily of same lengths) that are required so as to divide the rectangle into n subrectangles? Justify.

For example, in the adjacent figure, 3 segments are drawn to get 5 subrectangles and 3 is the minimum number.
Solution: It can be observed that there are three cases.
Case 1: Let $n = k^2$. Consider $(k-1)$ horizontal and $(k-1)$ vertical line segments. These $2(k-1)$ line segments will generate k^2 subrectangles. Note that this is a configuration with minimum number of line segments to divide the rectangle into k^2 subrectangles. [2]
Case 2: Let $k^2 + 1 \leq n \leq k^2 + k$. Consider a small horizontal line segment, which divides one of k^2 subrectangles into 2 subrectangles resulting into total $k^2 + 1$ subrectangles. Extending this small line segment, we can get up to $k^2 + k$ subrectangles. Hence, in this case the the minimum number of line segments required to divide the rectangle into n subrectangles is $2(k-1) + 1 = 2k - 1$. [3]
Case 3: Let $k^2 + k + 1 \leq n \leq k^2 + 2k + 1$. Applying the same procedure as given in case 2, but instead of horizontal line segment we need to take a vertical line segment and extend it as above. Thus, in this case the minimum number of line segments required to divide the rectangle into n subrectangles is $(2k-1)+1 = 2k$. [1]

4. Let $f : [0,1] \to (0,\infty)$ be a continuous function satisfying $\displaystyle\int_0^1 f(t)dt = \frac{1}{3}$. Show that there exists $c \in (0,1)$ such that $\displaystyle\int_0^c f(t)dt = c - \frac{1}{2}$.

Solution: Define $g(x) = \displaystyle\int_0^x f(t)dt + \frac{1}{2}$. [2]

Then $g : [0,1] \to [0,1]$ is a continuous function. By fixed point theorem, there exists $c \in (0,1)$ such that $g(c) = c$. Observe that $c \neq 0, 1$. Thus we get $\displaystyle\int_0^c f(t)dt = c - \frac{1}{2}$. [4]

5. Let $A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$. Show that there exist matrices X, Y such that $A = X^3 + Y^3$.

Solution: Note that $A^2 + 3A + 2I = 0$. [2]

Therefore $A(A^2 + 3A + 2I) = A^3 + 3A^2 + 2A = 0$. This implies that
$(A + I)^3 = A^3 + 3A^2 + 3A + I = A + I$. [2]

Hence $A = (A + I)^3 - I = (A + I)^3 + (-I)^3$.

Thus we get $X = A + I = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ and $Y = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. [2]

Part III

1. Let $f : (0,\infty) \to \mathbb{R}$ be a continuous function satisfying $f(1) = 5$ and $f\left(\dfrac{x}{x+1}\right) = f(x) + 2$ for all positive real numbers x.

a) Find $\lim_{x\to\infty} f(x)$.

b) Show that $\lim_{x\to 0^+} f(x) = \infty$.

c) Find one example of such a function. [12]

Solution: a) Note that as $x \to \infty$, $\dfrac{x}{x+1} \to 1$.

Hence $\lim_{x\to\infty} f(x) = f(1) - 2 = 5 - 2 = 3$. [2]

b) Observe that as $x \to 0^+$, $\dfrac{x}{x+1} \to 0$. If $\lim_{x\to 0^+} f(x) = L$, then $L = L + 2$. This implies that the limit is not finite. [1]

Define $g : [\frac{1}{n+1}, \frac{1}{n}] \to [\frac{1}{n+2}, \frac{1}{n+1}]$ as $g(x) = \dfrac{x}{x+1}$. Note that $g(\frac{1}{n+1}) = \frac{1}{n+2}$, $g(\frac{1}{n}) = \frac{1}{n+1}$ and $g'(x) = \frac{1}{(x+1)^2} > 0$. Therefore g is an increasing function and hence g is one-one and onto. For $x \in [\frac{1}{3}, \frac{1}{2}]$, there is some $t \in [\frac{1}{2}, 1]$ such that $f(x) = f(g(t)) = f(\frac{t}{t+1}) = f(t) + 2$. By induction, it can be proved that for $x \in [\frac{1}{n+1}, \frac{1}{n}]$, there is some $t \in [\frac{1}{2}, 1]$ such that $f(x) = f(t) + 2(n-1)$.

Let $M > 0$ be any real number. Since f is continuous on $[\frac{1}{2}, 1]$, it is bounded and attains its bounds. There exists $M_1 > 0$ such that $-M_1 < f(t) < M_1,\ \forall t \in [\frac{1}{2}, 1]$. Suppose $f(t_0) = \min f(t),\ t \in [\frac{1}{2}, 1]$. Therefore $f(t) \geq f(t_0),\ \forall t \in [\frac{1}{2}, 1]$. Choose $n_1 \in \mathbb{N}$ such that $f(t_0) + 2(n_1 - 1) > 0$. We can choose $n_2 \in \mathbb{N}$ such that

$n_2 > M + M_1 + 2$. Suppose $n_0 = \max\{n_1, n_2\}$ and $\delta = \frac{1}{n_0} > 0$. If $0 < x < \delta$, then $x \in [\frac{1}{n_0+1}, \frac{1}{n_0}]$ and $|f(x)| = |f(t) + 2(n_0 - 1)|,\ t \in [\frac{1}{2}, 1]$.
Now $f(t) + 2(n_0 - 1) \geq f(t_0) + 2(n_0 - 1) \geq f(t_0) + 2(n_1 - 1) > 0$.
Note that $|f(x)| = f(t) + 2(n_0 - 1) \geq -M_1 + 2(n_0 - 1) > -M_1 - 2 + n_0 \geq -M_1 - 2 + n_2 > M$.
Hence $\lim_{x \to 0^+} f(x) = \infty$. [6]

c) Example: $f(x) = 3 + \dfrac{2}{x}$. [3]

2. An $n \times n$ matrix $A = (a_{ij})$ is given. The sum of any n entries of A, whose any two entries lie on different rows and different columns, is the same.
a) Prove that there exist numbers $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ such that $a_{ij} = x_i + y_j$ for all i, j, $\ 1 \leq i, j \leq n$.
b) Prove that $\text{rank}(A) \leq 2$. [12]

Solution: a) Consider n entries situated on different rows and different columns $a_{ij_i}, i = 1, 2, \cdots, n$. Fix k and $l, 1 \leq k < l \leq n$ and replace a_{kj_k} and a_{lj_l} with a_{kj_l} and a_{lj_k} respectively. The new n entries are still situated on different rows and different columns. Since the sums of sets of n entries are equal, this implies
$a_{kj_k} + a_{lj_l} = a_{kj_l} + a_{lj_k}$. [*]
Now denote by $x_1, x_2, \cdots, x_n$ the entries in the first column and by $x_1, x_1+y_2, x_1+y_3, \cdots, x_1 + y_n$ the entries in the first row.

$$A = \begin{pmatrix} x_1 & x_1 + y_2 & x_1 + y_3 & \cdots & x_1 + y_n \\ x_2 & \cdots & & & \\ \vdots & & & & \\ x_n & \cdots & & & \end{pmatrix}$$

That is we have defined $x_k = a_{k1}$ for all $k, y_1 = 0$ and $y_k = a_{1k} - a_{k1}$ for all $k \geq 2$. Now $a_{ij} = x_i + y_j$ for all i, j with $i = 1$ or $j = 1$. Consider $i, j > 1$. From [*] we deduce that $a_{11} + a_{ij} = a_{1j} + a_{i1}$. Hence $x_1 + a_{ij} = x_i + x_1 + y_j$. This implies $a_{ij} = x_i + y_j$ for all i, j. [8]
b) Let A_j denote the j^{th} column of matrix A. Then

$$A_j = \begin{pmatrix} x_1 + y_j \\ x_2 + y_j \\ \vdots \\ x_n + y_j \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + y_j \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \alpha + y_j\beta, \text{ where } \alpha = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

That is $A_j \in \langle \alpha, \beta \rangle$, where $\langle \alpha, \beta \rangle$ denotes the linear span of α, β. This is true for every column of A. Hence $\text{rank}(A) \leq \dim\ \langle \alpha, \beta \rangle \leq 2$. [4]

3. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function. Let

$$J = \left\{ \frac{f(b) - f(a)}{b - a}\ :\ a, b \in I, a < b \right\}.$$

Show that a) J is an interval.
b) $J \subseteq f'(I)$ and $f'(I) - J$ contains at most two elements. [13]

Solution: a) Note that J is an interval if and only if for every $a, b \in J, a < b$, we have $(a, b) \subset J$. Let $u, v \in J, u < v$. Hence $u = \dfrac{f(b_1) - f(a_1)}{b_1 - a_1}, v = \dfrac{f(b_2) - f(a_2)}{b_2 - a_2}$. Suppose $p \in (u, v)$. Define a function Q on $[0, 1]$ as

$$Q(t) = \frac{f(tb_1 + (1-t)b_2) - f(ta_1 + (1-t)a_2)}{t(b_1 - a_1) + (1-t)(b_2 - a_2)}.$$

Since Q is a continuous function and $Q(0) = v, Q(1) = u$, we deduce that there exists t_0 such that $Q(t_0) = p$. Hence $p = \dfrac{f(b_0) - f(a_0)}{b_0 - a_0}$, where $b_0 = t_0 b_1 + (1-t_0) b_2$ and $a_0 = t_0 a_1 + (1 - t_0) a_2$. Thus $p \in J$. [6]

b) Using Lagrange's Mean Value Theorem, we get $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ for some $c \in (a, b)$. Hence $J \subseteq f'(I)$. [2]

Let $p \in I$ and $y_n = \dfrac{f(p + (1/n)) - f(p)}{1/n}$. If p is the right end point of an interval I, then choose $y_n = \dfrac{f(p) - f(p - (1/n))}{1/n}$. Note that y_n converges to $f'(p)$. Hence $f'(p) \in \overline{J}$. Therefore $f'(I) \subseteq \overline{J}$. Since $J \subseteq f'(I) \subseteq \overline{J}$ and J being an interval, the set $f'(I) - J$ contains at most two elements, which are the end points of J. [5]

4. Let q, n be positive integers such that $1 < q < n$ and $\gcd(q, n) = 1$.
a) Show that there exist unique integers k, r such that $n = kq - r,\ 0 \le r < q$.
b) Show that there exists a unique positive integer m and unique integers $b_1, b_2, \cdots, b_m$ all ≥ 2 satisfying $\dfrac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \ddots - \cfrac{1}{b_{m-1} - \cfrac{1}{b_m}}}}$.
c) If $b_j > 2$ for some j, then show that $\displaystyle\sum_{i=1}^{m} (b_i - 2) < 2(n - q - 1)$. [13]

Solution:

(a) By division algorithm, there exists unique $k_1, r_1 \in \mathbb{N}$ such that $n = k_1 q + r_1$, where $0 \le r_1 < q$. Further, $\gcd(q, n) = 1 \implies r_1 > 0$.
Thus, we get, $n = (k_1 + 1)q - (q - r_1)$. Since, $r_1 < q$, we have, $r = q - r_1 > 0,\ r < q$ and let $k = k_1 + 1 \in \mathbb{N}$.
Hence, there exists unique $k, r \in \mathbb{N}$ such that $n = kq - r$, where $0 < r < q$. [3]

(b) By (a), we get $k, r \in \mathbb{N}$ such that $n = kq - r$ and $0 < r < q$.
Thus, $\dfrac{n}{q} = k - \dfrac{r}{q}$. Let $b_1 = k$. Now, $n > q \implies k \ge 2 \implies b_1 \ge 2$.

Observe : $\dfrac{n}{q} = b_1 - \dfrac{1}{q/r}$. Now, $q > r$; $\gcd(q, r) = 1$. Let $r_1 = r$. Hence, by applying the above procedure to q and r_1, we get,

$$\frac{q}{r_1} = b_2 - \frac{1}{r_1/r_2}$$

where, $b_2 \geq 2$ and $r_2 < r_1$.
Repeating these steps, we get, $r_1, r_2, \ldots, r_m$ such that $0 < r_m < r_{m-1} < r_{m-2} < \cdots < r_2 < r_1$. The process ends after finitely many steps(say m steps), with $r_m = 1$.
Now, we prove that, whenever, in the above expression, $b_i \geq 2$, for each $i = 1, 2, \ldots, m$ then such $b_i s$ are unique. ....(I)
In fact, we prove that if $b_i \geq 2$, for all $i = 1, 2, \ldots, m$ then $b_1 = \left\lfloor \dfrac{n}{q} \right\rfloor + 1$
Case (i) if $b_1 > \left\lfloor \dfrac{n}{q} \right\rfloor + 1$.
In this case, we can write: $\dfrac{n}{q} = b_1 - \dfrac{r}{q}$, for some $r > q$.
Thus, $\dfrac{n}{q} = b_1 - \dfrac{1}{q/r} = b_1 - \dfrac{1}{b_2 - s}$; where $b_2 - s = \dfrac{q}{r}$, giving $s = b_2 - \dfrac{q}{r}$
Now, we have, $r > q$ and $b_2 \geq 2$. Hence, $s > 1$.
Thus, $\dfrac{n}{q} = b_1 - \dfrac{1}{b_2 - s}$, where $s > 1$ ......(II)
The above steps can be repeated to get $s_1 > 1$ such that
$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - s_1}}$$
Hence, the process does not end after finitely many steps. This is a contradiction, since, we have finitely many b_is.
Case (ii) If $b_1 < \left\lfloor \dfrac{n}{q} \right\rfloor + 1$.
In this case, we can write $\dfrac{n}{q} = b_1 + \dfrac{r}{q}$, for some $r > 0$.
Thus, $\dfrac{n}{q} = b_1 - \dfrac{1}{-q/r} = b_1 - \dfrac{1}{b_2 - s}$; where $b_2 - s = -\dfrac{q}{r}$, giving $s = b_2 + \dfrac{q}{r}$
Now, we have $r > 0,\ q > 0$ and $b_2 \geq 2$. Thus, $s \geq 2 > 1$
Hence, we have, $\dfrac{n}{q} = b_1 - \dfrac{1}{b_2 - s}$ with $s > 1$. Again as in case (i), the process does not end after finitely many steps.
Hence, we must have $b_1 = \left\lfloor \dfrac{n}{q} \right\rfloor + 1$.
Now by replacing n by q and q by r, we get b_2 is unique. Similarly we get that all b_is are unique. [5]

(c) To prove that if $b_j > 2$ for some $j \in \{1, 2, \ldots, m\}$ then
$\displaystyle\sum_{i=1}^{m} (b_i - 2) < 2(n - q - 1)$.

We apply second principle of induction on n.
Observe that for $n = 2,\ n = 3,\ n = 4$, there is no value of q satisfying all the conditions.
To prove for $n = 5$:
Case (i) $q = 2$. We have $\frac{5}{2} = 3 - \frac{1}{2}$ and by (I), such expression is unique.
Case (ii) $q = 3$. We have $\frac{5}{3} = 2 - \frac{1}{3}$ and by (I), such expression is unique.
Let $n > 5$. Assume the result is true for all $k \leq n$.
To prove the result for $n + 1$.
Let $1 < q < n + 1$ and $\gcd(n + 1, q) = 1$

$$\frac{n+1}{q} = b_1 - \frac{1}{q/r} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 -}} \ddots - \frac{1}{b_{m-1} - \frac{1}{b_m}}$$

By uniqueness of b_is, we have, $\frac{q}{r} = b_2 - \cfrac{1}{b_3 -} \ddots - \frac{1}{b_{m-1} - \frac{1}{b_m}}$. If no $b_i > 2$ for $i = 2, 3, \ldots, m$ then we get, $b_i = 2$, for each $i = 2, 3, \ldots, m$. So, we must have $b_1 > 2$.
In this case, we get $\frac{n+1}{q} = b_1 - \cfrac{1}{2 - \cfrac{1}{2 -}} \ddots - \frac{1}{2 - \frac{1}{2}}$.
Observe that $\frac{1}{2 - \frac{k-1}{k}} = \frac{k}{k+1}$. Hence $\frac{n+1}{q} = b_1 - \frac{k-1}{k}$.
Now $n + 1 = b_1 q - \frac{(k-1)}{k} q$. Thus $k|q$. Let $q = km$. Then we have
$n + 1 = b_1 q - (k - 1)m = b_1 q - km + m = b_1 q - q + m$.
Therefore $n - q = b_1 q - 2q + m - 1 = q(b_1 - 2) + (m - 1)$.
Thus we get $2(n + 1 - q - 1) = 2(n - q) = 2q(b_1 - 2) + 2(m - 1) > b_1 - 2$ as required.

Now, if $b_i > 2$, for some $i = 2, 3, 4 \ldots, m$ then applying induction hypothesis to q, we get,

$$\sum_{i=2}^{m} (b_i - 2) < 2(q - r - 1)$$

We need to prove that $\sum_{i=1}^{m} (b_i - 2) < 2(n + 1 - q - 1)$.
It is enough to prove that $2(q - r - 1) + b_1 - 2 \leq 2(n + 1 - q - 1)$
That is to prove that $2q - 2r - 2 + b_1 - 2 \leq 2(b_1 q - r - q - 1)$,
which is equivalent to $b_1 - 2 \leq 2q(b_1 - 2)$ which holds. Hence, the result is true for $n + 1$. Thus by second principal of induction, the result is true for all $n \geq 5$. [5]