MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 12/01/2020 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let A be a non-empty subset of real numbers and $f : A \to A$ be a function such that $f(f(x)) = x$ for all $x \in A$. Then $f(x)$ is
A) a bijection B) one-one but not onto
C) onto but not one-one D) neither one-one nor onto.

2. If $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x+y) = f(xy)$ for all $x, y \in \mathbb{R}$ and $f(3/4) = 3/4$, then $f(9/16) =$
A) 9/16 B) 0 C) 3/2 D) 3/4.

3. The area enclosed between the curves $y = \sin^2 x$ and $y = \cos^2 x$ in the interval $0 \leq x \leq \pi/2$ is
A) 2 B) 1/2 C) 1 D) 3/4.

4. The number of ordered pairs (m, n) of all integers satisfying $\dfrac{m}{12} = \dfrac{12}{n}$ is
A) 15 B) 30 C) 12 D) 10.

5. Suppose $2 \log x + \log y = x - y$. Then the equation of the tangent line to the graph of this equation at the point $(1, 1)$ is
A) $x + 2y = 3$ B) $x - 2y = 3$ C) $2x + y = 3$ D) $2x - y = 3$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sin[x]$, where $[x]$ denotes the greatest integer less than or equal to x. Then
A) f is a 2π-periodic function B) f is a π-periodic function
C) f is a 1-periodic function D) f is not a periodic function.

7. For how many integers a with $1 \leq a \leq 100$, a^a is a square?
A) 50 B) 51 C) 55 D) 56.

8. $\displaystyle\lim_{x \to 0} x \left[\frac{1}{x}\right]$
A) 0 B) 1 C) -1 D) does not exist.

9. If α and β are the roots of $x^2 + 3x + 1$ then $\left(\dfrac{\alpha}{\beta+1}\right)^2 + \left(\dfrac{\beta}{\alpha+1}\right)^2$ equals
A) 19 B) 18 C) 20 D) 17.

10. The equation $z^3 + iz - 1 = 0$ has
A) no real root B) exactly one real root
C) three real roots D) exactly two real roots.

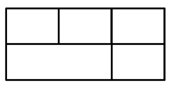
Part II

N.B. Each question in Part II carries 6 marks.

1. Let $a_1, a_2, \cdots$ be a sequence of natural numbers. Let (a, b) denote the greatest common divisor (gcd) of a and b. If $(a_m, a_n) = (m, n)$ for all $m \neq n$, prove that $a_n = n$ for all $n \in \mathbb{N}$.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(z)f(iz) = z^2$ for all $z \in \mathbb{C}$. Prove that $f(z) + f(-z) = 0$ for all $z \in \mathbb{C}$. Find such a function.

3. Let n be a positive integer. Line segments can be drawn parallel to edges of a given rectangle. What is the minimum number of line segments (not necessarily of same lengths) that are required so as to divide the rectangle into n subrectangles? Justify.

   For example, in the adjacent figure, 3 segments are drawn to get 5 subrectangles and 3 is the minimum number.

4. Let $f : [0, 1] \to (0, \infty)$ be a continuous function satisfying $\displaystyle\int_0^1 f(t)dt = \frac{1}{3}$. Show that there exists $c \in (0, 1)$ such that $\displaystyle\int_0^c f(t)dt = c - \frac{1}{2}$.

5. Let $A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$. Show that there exist matrices X, Y such that $A = X^3 + Y^3$.

Part III

1. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function satisfying $f(1) = 5$ and $f\left(\frac{x}{x+1}\right) = f(x) + 2$ for all positive real numbers x.
   a) Find $\lim\limits_{x \to \infty} f(x)$.
   b) Show that $\lim\limits_{x \to 0^+} f(x) = \infty$.
   c) Find one example of such a function. [12]

2. An $n \times n$ matrix $A = (a_{ij})$ is given. The sum of any n entries of A, whose any two entries lie on different rows and different columns, is the same.
   a) Prove that there exist numbers $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ such that $a_{ij} = x_i + y_j$ for all i, j, $1 \leq i, j \leq n$.
   b) Prove that $\operatorname{rank}(A) \leq 2$. [12]

3. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function. Let
$$J = \left\{ \frac{f(b) - f(a)}{b - a} \ : \ a, b \in I, a < b \right\}.$$
   Show that a) J is an interval.
   b) $J \subseteq f'(I)$ and $f'(I) - J$ contains at most two elements. [13]

4. Let q, n be positive integers such that $1 < q < n$ and $\gcd(q, n) = 1$.
   a) Show that there exist unique integers k, r such that $n = kq - r$, $0 \leq r < q$.
   b) Show that there exists a unique positive integer m and unique integers $b_1, b_2, \cdots, b_m$ all ≥ 2 satisfying
$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \ddots - \cfrac{1}{b_{m-1} - \frac{1}{b_m}}}}.$$
   c) If $b_j > 2$ for some j, then show that $\displaystyle\sum_{i=1}^{m} (b_i - 2) < 2(n - q - 1)$. [13]