MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 08/01/2017 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The number $\sqrt{2}e^{i\pi}$ is:
   A) a rational number.
   B) an irrational number.
   C) a purely imaginary number.
   D) a complex number of the type $a + ib$ where a, b are non-zero real numbers.

2. Let P= $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The rank of P^4 is: A) 1 B) 2 C) 3 D) 4.

3. Let $y_1(x)$ and $y_2(x)$ be the solutions of the differentiable equation $\frac{dy}{dx} = y + 17$ with initial conditions $y_1(0) = 0, y_2(0) = 1$. Which of the following statements is true?
   A) y_1 and y_2 will never intersect.
   B) y_1 and y_2 will intersect at $x = e$.
   C) y_1 and y_2 will intersect at $x = 17$.
   D) y_1 and y_2 will intersect at $x = 1$.

4. Suppose f and g are differentiable functions and $h(x) = f(x)g(x)$. Let $h(1) = 24$, $g(1) = 6$, $f'(1) = -2$, $h'(1) = 20$. Then the value of $g'(1)$ is
   A) 8 B) 4 C) 2 D) 16.

5. In how many regions is the plane divided when the following equations are graphed, not considering the axes? $y = x^2,\ y = 2^x$
   A) 3 B) 4 C) 5 D) 6.

6. For $0 \leq x < 2\pi$, the number of solutions of the equation $\sin^2 x + 3\sin x\cos x + 2\cos^2 x = 0$ is
   A) 1 B) 2 C) 3 D) 4.

7. The minimum value of the function $f(x) = x^x,\ x \in (0, \infty)$ is
   A) $\left(\frac{1}{10}\right)^{\frac{1}{10}}$ B) $10^{\frac{1}{10}}$ C) $\frac{1}{e}$ D) $\left(\frac{1}{e}\right)^{\frac{1}{e}}$.

8. Let f be a twice differentiable function on $\mathbb{R}$. Also $f''(x) > 0$ for all $x \in \mathbb{R}$. Which of the following statements is true?
   A) $f(x) = 0$ has exactly two solutions on $\mathbb{R}$.
   B) $f(x) = 0$ has a positive solution if $f(0) = 0$ and $f'(0) = 0$.
   C) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) > 0$.
   D) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) < 0$.

9. If $x^2+x+1=0$, then the value of $\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\ldots+\left(x^{27}+\frac{1}{x^{27}}\right)^2$ is

A) 27 B) 54 C) 0 D) -27.

10. Let M= $\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$. Then $M^{2017}=$

A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ B) $\begin{pmatrix} -2^{2017} & -1 \\ 3^{2017} & 1 \end{pmatrix}$ C) $\begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$ D) $\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let a,b,c be real numbers such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $abc=1$. Prove that at least one of a,b,c is 1.

2. Let $c_1,c_2,\ldots,c_9$ be the zeros of the polynomial $z^9-6z^7+12z^6+18z^4-24z^3+30z^2-z+2017$. If $S(z)=\sum_{k=1}^{9}|z-c_k|^2$, then prove that $S(z)$ is constant on the circle $|z|=100$.

3. Let f be a monic polynomial with real coefficients. Let $\lim_{x\to\infty} f''(x)=\lim_{x\to\infty} f\left(\frac{1}{x}\right)$ and $f(x)\geq f(1)$ for all $x\in\mathbb{R}$. Find f.

4. Call a set of integers *non−isolated* if for every $a\in A$ at least one of the numbers $a-1$ and $a+1$ also belongs to A. Prove that the number of 5−element *non−isolated* subsets of $\{1,2,\ldots,n\}$ is $(n-4)^2$.

5. Find all positive integers n for which a permutation $a_1,a_2,\ldots,a_n$ of $\{1,2,\ldots,n\}$ can be found such that $0,a_1,a_1+a_2,a_1+a_2+a_3,\ldots,a_1+a_2+\ldots+a_n$ leave distinct remainders modulo $n+1$.

Part III

1. Do there exist 100 lines in the plane, no three concurrent such that they intersect exactly in 2017 points? [12]

2. On the parabola $y=x^2$, a sequence of points $P_n(x_n,y_n)$ is selected recursively where the points P_1,P_2 are arbitrarily selected distinct points. Having selected P_n, tangents drawn at P_{n-1} and P_n meet at say Q_n. Suppose P_{n+1} is the point of intersection of $y=x^2$ and the line passing through Q_n parallel to Y-axis. Under what conditions on P_1,P_2

   (a) both the sequences $\{x_n\}$ and $\{y_n\}$ converge?

   (b) $\{x_n\}$ and $\{y_n\}$ both converge to 0? [13]

3. (a) Show that there does not exist a 3-digit number A such that 10^3A+A is a perfect square.

   (b) Show that there exists an n-digit ($n>3$) number A such that 10^nA+A is a perfect square. [12]

4. For $n\times n$ matrices A,B, let $C=AB-BA$. If C commutes with both A and B, then

   (a) Show that $AB^k-B^kA=kB^{k-1}C$ for every positive integer k.

   (b) Show that there exists a positive integer m such that $C^m=0$. [13]