MADHAVA MATHEMATICS COMPETITION, 8 January 2012
Solutions and Scheme of Marking

Part I

N.B. Each question in Part I carries 2 marks.

1. Let n be a fixed positive integer. The value of k for which $\int_1^k x^{n-1}dx = \frac{1}{n}$ is
(a) 0 (b) 2^n (c) $(\frac{2}{n})^{\frac{1}{n}}$ (d) $2^{\frac{1}{n}}$
Solution: (d)
$\int_1^k x^{n-1}dx = [\frac{x^n}{n}]_1^k = \frac{k^n}{n} - \frac{1}{n} = \frac{1}{n}$.
Therefore $\frac{k^n}{n} = \frac{2}{n}$. Therefore $k^n = 2$. Hence $k = 2^{\frac{1}{n}}$.

2. Let $S = \{a, b, c\}$, $T = \{1, 2\}$. If m denotes the number of one-one functions and n denotes the number of onto functions from S to T, then the values of m and n respectively are
(a) 6,0 (b) 0,6 (c) 5,6 (d) 0,8.
Solution: (b)
There can not be a one-one function from S to T because the number of elements in S is bigger than the number of elements in T. The total number of functions from S to T is 2^3. Since the constant function from S to T is not onto, the number of onto functions is $2^3 - 2 = 6$.

3. In the binary system, $\frac{1}{2}$ can be written as
(a) $0.01111\cdots$ (b) $0.01000\cdots$ (c) $0.0101\cdots$ (d) None of these
Solution: (a)
Observe that $\sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}$.

4. For the equation $|x|^2 + |x| - 6 = 0$
(a) there is only one root. (b) the sum of the roots is -1.
(c) the sum of the roots is 0. (d) the product of the roots is -6.
Solution: (c)
$|x|^2 + |x| - 6 = (|x| + 3)(|x| - 2) = 0$. Therefore roots are $|x| = 2$ i.e. $x = \pm 2$. Hence the sum of the roots is 0.

5. The value of $\lim_{n\to\infty} \frac{1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!}{(n+1)!}$
(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) does not exist.
Solution: (a)
Observe that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (2-1) \cdot 1! + (3-1) \cdot 2! + \cdots + ((n+1)-1) \cdot n! = (2! - 1!) + (3! - 2!) + \cdots + (n+1)! - n! = (n+1)! - 1$.
Therefore $\lim_{n\to\infty} \frac{1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!}{(n+1)!} = \lim_{n\to\infty} \frac{(n+1)! - 1}{(n+1)!} = 1$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f'(x) > g'(x)$ for every x. Then the graphs $y = f(x)$ and $y = g(x)$
(a) intersect exactly once. (b) intersect at most once.
(c) do not intersect. (d) could intersect more than once.

Solution: (b)

Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) > 0$. Therefore h is strictly increasing. If the graph of h intersects $X-$axis, then $f(x) = g(x)$ at exactly one point. It is possible that the graphs of f and g do not intersect. For example, $f(x) = e^x$, $g(x) = 0$. Therefore the graphs $y = f(x)$ and $y = g(x)$ intersect at most once.

7. The function $|x|^3$ is

(a) differentiable twice but not thrice at 0. (b) not differentiable at 0.
(c) three times differentiable at 0. (d) differentiable only once at 0.

Solution: (a)

Observe that $f'(0) = 0$ and $f''(0) = 0$.

$f'''(0) = \lim_{h \to 0} \dfrac{f''(h)}{h}$. Then left limit is -6 and right limit is 6. Therefore limit does not exist.

Hence f is differentiable twice but not thrice at 0.

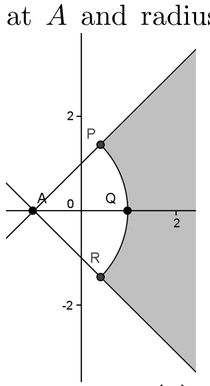
8. Let A be the $n \times n$ matrix $(n \geq 2)$, whose $(i,j)^{th}$ entry is $i + j$ for all $i, j = 1, 2, \cdots, n$. The rank of A is

(a) 2 (b) 1 (c) n (d) $n - 1$.

Solution: (a)

$$\text{rank}\begin{pmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & \dots & 2n \end{pmatrix} = \text{rank}\begin{pmatrix} 2 & 3 & \dots & n+1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = \text{rank}\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = 2.$$

9. Let $A = -1 + 0i$ be the point in the complex plane. Let PQR be an arc with centre at A and radius 2. If $P = -1 + \sqrt{2} + \sqrt{2}i, Q = 1 + 0i$ and $R = -1 + \sqrt{2} - \sqrt{2}i$



then the shaded region is given by

(a) $|z + 1| < 2, |\arg(z + 1)| < \frac{\pi}{2}$
(b) $|z - 1| < 2, |\arg(z - 1)| < \frac{\pi}{2}$
(c) $|z + 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$
(d) $|z - 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$

Solution: (c)

Since PQR is an arc with centre at -1 and radius 2, it satisfies the equation $|z+1| = 2$. Therefore any point in the the shaded region satisfies the condition $|z + 1| > 2$.

10. The solution of $\dfrac{dy}{dx} = a^{x+y}$ is

(a) $a^x - a^{-y} = c$ (b) $a^{-x} + a^{-y} = c$ (c) $a^{-x} - a^y = c$ (d) $a^x + a^{-y} = c$

Solution: (d)

$\dfrac{dy}{dx} = a^{x+y}$. Therefore $a^{-y}dy = a^x dx$. Integrating we get, $a^x + a^{-y} = c$.

Part II

N.B. Each question in Part II carries 5 marks.

1. Let $f : [0,4] \to [3,9]$ be a continuous function. Show that there exists x_0 such that $f(x_0) = \dfrac{3x_0+6}{2}$.

   Solution:

   Define a function $g : [3,9] \to [0,4]$ as $g(x) = \dfrac{2x-6}{3}$. [2]

   Then $g \circ f : [0,4] \to [0,4]$ is a continuous function. By fixed point theorem, there exists x_0 such that $g(f(x_0)) = x_0$. i.e. $\dfrac{2f(x_0)-6}{3} = x_0$. Hence $f(x_0) = \dfrac{3x_0+6}{2}$. [3]

2. Suppose a, b, c are all real numbers such that $a+b+c > 0$, $abc > 0$ and $ab+bc+ac > 0$. Show that a, b, c are all positive.

   Solution:

   Let $a+b+c = u$,

   $ab+bc+ca = v$

   and $abc = w$

   Then a, b, c are the roots of the equation $x^3 - ux^2 + vx - w = 0$ [2]

   Now $u, v, w > 0$. If $x \leq 0$, LHS is -ve.

   $\Rightarrow$ all real roots are positive. [3]

   Hence a, b, c are all positive.

3. Let f be a continuous function on $[0,2]$ and twice differentiable on $(0,2)$. If $f(0) = 0, f(1) = 1$ and $f(2) = 2$, then show that there exists x_0 such that $f''(x_0) = 0$.

   Solution:

   Given that $f(0) = 0, f(1) = 1$ and $f(2) = 2$.

   By Mean Value Theorem, there exist $c \in (0,1)$ and $d \in (1,2)$ such that $\dfrac{f(1)-f(0)}{1-0} = 1 = f'(c)$ and $\dfrac{f(2)-f(1)}{2-1} = 1 = f'(d)$. [3]

   Therefore $f'(c) = f'(d)$.

   Applying Rolle's Theorem, there exists $x_0 \in (c,d)$, such that $f''(x_0) = 0$. [2]

4. Integers $1, 2, \cdots, n$ are placed in such a way that each value is either bigger than all preceding values or smaller than all preceding values. In how many ways this can be done? (For example in case of $n = 5$, 3 2 4 1 5 is valid and 3 2 5 1 4 is not valid.)

   Solution:

   The last term must be 1 or n and each subsequent term from the right must be either largest or smallest of those left. The first term is uniquely determined. [2]

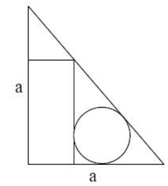
   Thus every position except 1st has two choices and so number of such sequences is 2^{n-1}.

   [3]

Part III

N.B. Each question in Part III carries 12 marks.

1. Consider an isosceles right triangle with legs of fixed length a.



Inscribe a rectangle and a circle inside the triangle as indicated in the figure. Find the dimensions of the rectangle and the radius of the circle which make the total area of the rectangle and circle maximum.

Solution: Let r be the radius of a circle and x be the smaller side of a rectangle. We want to maximize $f(x) = (a-x)x + \pi r^2$.
Since we have the relationship

$$\frac{1}{2}(a-x)(a-x) = \frac{1}{2}r\sqrt{2}(a-x) + \frac{1}{2}r(a-x) + \frac{1}{2}r(a-x),$$

we see that, $r = \dfrac{(a-x)(a-x)}{(a-x)+(a-x)+\sqrt{2}(a-x)} = \dfrac{(a-x)}{2+\sqrt{2}}$. Then we want to maximize
$f(x) = (a-x)x + \pi\dfrac{(a-x)^2}{(2+\sqrt{2})^2}$. [7]
where $0 \le x \le a$. Letting $A = \dfrac{\pi}{(2+\sqrt{2})^2}$, we can write
$f(x) = x^2(A-1) + x(a-2aA) + Aa^2$.
Differentiating f with respect to x we get, $f'(x) = 2x(A-1) + a(1-2A)$.
which has critical point at $x = \dfrac{a(1-2A)}{2(A-1)}$. Further this gives maximum since $A < 1$ gives us $f''(x) = 2(A-1) < 0$. Hence $r = \dfrac{a(4A-3)}{2(2+\sqrt{2})(A-2)}$. [5]

2. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, one-one function. If there exists a positive integer n such that $f^n(x) = x$, for every $x \in \mathbb{R}$, then prove that either $f(x) = x$ or $f^2(x) = x$. (Note that $f^n(x) = f(f^{n-1}(x))$.)

Solution: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous and one-one function. Then f is either strictly increasing or strictly decreasing. [4]
We now prove that if f is strictly increasing, then $f(x) = x$ and if f is strictly decreasing, then $f^2(x) = x$.
Case-I : Suppose f is strictly increasing and there exists x_0 such that $f(x_0) \neq x_0$.
Let $f(x_0) > x_0$. Since f is strictly increasing, $f^2(x_0) > f(x_0) > x_0$. Therefore $f^n(x_0) > x_0$, which is a contradiction.
Let $f(x_0) < x_0$. By similar arguments, we get a contradiction. [3]
Case-II : Suppose f is strictly decreasing. Then f^2 is strictly increasing. Observe that $f^n(x) = x$, for every $x \in \mathbb{R}$ implies $f^{2n}(x) = x$ for every $x \in \mathbb{R}$. Therefore $(f^2)^n(x) = x$, for every $x \in \mathbb{R}$. By applying case-I to f^2, we get $f^2(x) = x$. [5]

3. Consider $f(x) = \dfrac{x^3}{6} + \dfrac{x^2}{2} + \dfrac{x}{3} + 1$. Prove that $f(x)$ is an integer whenever x is an integer. Determine with justification, conditions on real numbers a, b, c and d so that $ax^3 + bx^2 + cx + d$ is an integer for all integers x.

Solution: Note that $f(x) = \frac{1}{6}(x^3 + 3x^2 + 2x + 6) = \frac{1}{6}(x^2 + 2)(x + 3)$. Let x be any integer. Then

$$3|x \;\Rightarrow\; 3|(x+3), \quad \text{and}$$
$$x = 3k \pm 1 \;\Rightarrow\; x^2 = 9k^2 \pm 6k + 1 \;\Rightarrow\; x^2 + 2 = 9k^2 \pm 6k + 3 \;\Rightarrow\; 3|(x^2+2).$$

So, in all cases, $3|(x^2 + 2)(x + 3)$. Secondly,

$$x = 2k \;\Rightarrow\; x^2 + 2 = 4k^2 + 2 \Rightarrow\; 2|(x^2+2), \quad \text{and}$$
$$x = 2k - 1 \;\Rightarrow\; x + 3 = 2k + 2 \;\Rightarrow\; 2|(x+3).$$

Thus, in both cases, $2|(x^2 + 2)(x + 3)$. Therefore, since 2 and 3 are co-prime,

$6 = (2 \times 3)|(x^2 + 2)(x + 3)$. Therefore, $f(x)$ is an integer. [3]

Next, let $g(x) = ax^3 + bx^2 + cx + d$. We will show that $g(x)$ is an integer whenever x is an integer, if and only if

(i) d is an integer and there exist integers A, B, C such that

(ii) $a = A/6, \quad b = B/2, \quad c = C/6$ and such that

(iii) $3|(A + C)$ and (iv) $2|(A + B + C)$.

First, suppose that $g(x)$ is an integer whenever x is an integer. Then $g(0) = d \;\Rightarrow\; d$ is an integer. [1]

So, since $g(1) = a + b + c + d$ and $g(-1) = -a + b - c + d$, we see that $u = a + b + c$ and $v = -a + b - c$ are integers. Hence $2b = u + v = B$, say, is an integer [1]

and $2(a+c) = u - v = w$, say, is an integer. Again, $g(2) = 8a + 4b + 2c + d$ is an integer, so that $6a + 2B + w$ is an integer. [1]

Hence $6a = A$, say, is an integer. Thus $6a + 6c = 3w$ is an integer. Hence $6c = C$, say, is an integer. This proves (i) and (ii). Also, then $g(x) = \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + \frac{1}{6}Cx + d$ $= \frac{1}{6}(Ax^3 + 3Bx^2 + Cx + 6d)$. [2]

Since $w = 2(A/6 + C/6) = (A + C)/3$, is an integer, we see that $3|(A + C)$. Finally, $g(3) = \frac{1}{6}(27A + 9B + 3C + 6d) = \frac{1}{2}(9A + 3B + C + 2d)$ is an integer. Hence $2|[(A + B + C) + 8A + 2B + 2d]$ so that $2|(A + B + C)$. So, (iii) and (iv) hold. [2]

Conversely, suppose that the conditions hold for $g(x)$. Then $g(x) = \frac{1}{6}h(x)$ where $h(x) = Ax^3 + 3Bx^2 + Cx + 6d$. Let x be any integer. Then $h(x) = x(Ax^2 + 3Bx + C) + 6d$ is an integer. Clearly, $3|x \;\Rightarrow\; 3|h(x)$. Next, let $x \equiv \pm 1$ modulo 3. Then modulo 3, $x^2 \equiv 1$ so that $h(x) \equiv \pm(A + C) \equiv 0$ by (iii). So, in all cases, $3|h(x)$. Also, clearly, $2|x \;\Rightarrow\; 2|h(x)$. Next, let $x \equiv 1$ modulo 2. Then modulo 2, $x^2 \equiv 1$ so that $h(x) \equiv (A + B + C) \equiv 0$ by (iv). So, in both cases, $2|h(x)$. So, $6|h(x)$. Therefore, $g(x)$ is an integer. [4]

4. Suppose $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & b & d \\ 1 & b & d+1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $U = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$. Find conditions on A and U such that the system $AX = U$ has no solution.

Solution: We use the result that the system $AX = U$ is *inconsistent* if and only if the ranks of the matrices A and $[A|U]$ are *not* equal. We apply elementary row transformations to the system $AX = U$ to reduce it to the form $A_1X = U_1$ where A_1 is lower triangular thus:

$$\left.\begin{array}{ccc|c} 2 & 1 & 0 & f \\ 1 & b & d & g \\ 1 & b & d+1 & h \end{array}\right.$$

$$\text{Apply } R_3 \to R_3 - R_2 : \quad \left.\begin{array}{ccc|c} 2 & 1 & 0 & f \\ 1 & b & d & g \\ 0 & 0 & 1 & h-g \end{array}\right.$$

$$\text{Apply } R_2 \to R_2 - \tfrac{1}{2}R_1 : \quad \left.\begin{array}{ccc|c} 2 & 1 & 0 & f \\ 0 & b-\frac{1}{2} & d & g-\frac{1}{2}f \\ 0 & 0 & 1 & h-g \end{array}\right.$$

So, if $b \neq \frac{1}{2}$, then A is invertible and the system has a unique solution for any matrix U. [6]

So, let $b = 1/2$. Apply $R_2 \to R_2 - dR_3$. Then

$$\left.\begin{array}{ccc|c} 2 & 1 & 0 & f \\ 0 & 0 & 0 & g-\frac{1}{2}f-d(h-g) \\ 0 & 0 & 1 & h-g \end{array}\right.$$

Thus rank of A_1 is 2 and rank of $[A_1|U_1]$ is 3 if and only if $g - \frac{1}{2}f - d(h-g) \neq 0$. Hence the system is not consistent if and only if $b = \frac{1}{2}$ and $g - \frac{1}{2}f - d(h-g) \neq 0$. [6]

5. Let A and B be finite subsets of the set of integers. Show that $|A + B| \geq |A| + |B| - 1$. When does equality hold?
(Here $A + B = \{x + y : x \in A,\ y \in B\}$. Also, $|S|$ denotes the number of elements in the set S.)

Solution: We first observe that the cardinalities of A, B and $A + B$ remain invariant under translation of A or B by any integer. [3]

We now translate A so as to have $\max A = 0$ and translate B so as to have $\min B = 0$. [3]

Thus now A has all non-positive integers, B has all non-negative integers and $A \cap B = \{0\}$. Hence $|A \cup B| = |A| + |B| - 1$. As both A and B contain 0, clearly $A \cup B \subset A + B$. So $|A \cup B| \leq |A + B|$ and hence $|A| + |B| - 1 \leq |A + B|$. [2]

Claim : Equality holds i.e $A \cup B \supset A + B$ if and only if $A \cup B$ is an arithmetic progression.

(I) Let $A \cup B$ be an arithmetic progression with common difference d. For $a \in A$ and $b \in B$, there exist $k \leq 0, l \geq 0$ such that $a = kd$ and $b = ld$. Then $a + b = (k + l)d$. So $a + b \in A \cup B$ since $k \leq k + l \leq l$.

(II)Given that $A \cup B = A + B$. Let $A = \{0 > a_1 > a_2 > \cdots > a_n\}$ and $B = \{0 < b_1 < b_2 < \cdots < b_m\}$. Now observe that $|a_1| = |b_1|$ because $a_1 < a_1 + b_1 < b_1$ and $a_1 + b_1 \in A \cup B$ so that we must have $a_1 + b_1 = 0$. Now by a similar argument, $a_2 + b_1 = a_1$. It is easy to see that, $a_{j+1} + b_1 = a_j$, for all j. So A is an A. P. with common difference b_1. Similarly B is an A. P. with common difference b_1. Thus $A \cup B$ is an arithmetic progression with common difference b_1. [4]