MADHAVA MATHEMATICS COMPETITION, January 4, 2015

Part I

N.B. Each question in Part I carries 2 marks.

1. How many five digit positive integers that are divisible by 3 can be formed using the digits $0, 1, 2, 3, 4$ and 5, without any of the digits getting repeated?

A) 216 B) 96 C) 120 D) 625 .

Answer: A) Numbers of the form $abcde$ with distinct digits from the set $\{0, 1, 2, 3, 4, 5\}$ such that $a \neq 0$ and $3 \mid (a + b + c + d + e)$. Since $3|(1 + \cdots + 5) = 15$, there are $5! = 120$ such numbers with no digit zero. If 0 is included, then 3 must be excluded; so for $a = 1$, $4!$ numbers like 10245; for $a = 2$, $4!$ numbers like 20145; for $a = 4$, $4!$ numbers like 40245; for $a = 5$, $4!$ numbers like 50124. So $120 + 96 = 216$ in all.

2. If $I = \displaystyle\int_0^1 \frac{1}{1 + x^8}\, dx$, then

A) $I < \dfrac{1}{2}$ B) $I < \dfrac{\pi}{4}$ C) $I > \dfrac{\pi}{4}$ D) $I = \dfrac{\pi}{4}$.

Answer: C) Note that $0 < x < 1 \Rightarrow x^8 < x^2 \Rightarrow 1 + x^8 < 1 + x^2 \Rightarrow 1/(1 + x^8) > 1/(1 + x^2)$. So $I > \int_0^1 1/(1 + x^2)dx = \pi/4$.

3. Find a and b so that $y = ax + b$ is a tangent line to the curve $y = x^2 + 3x + 2$ at $x = 3$.

A) $a = 9, b = -7$ B) $a = 3, b = -2$ C) $a = -9, b = 7$ D) $a = -3, b = 2$.

Answer: A) Since for $f(x) = x^2 + 3x + 2$, $f'(x) = 2x + 3$, the tangent at $x = 3$ is $y - f(3) = f'(3)(x - 3)$ i.e. $y - 20 = 9(x - 3)$ or $y = 9x - 7$.

4. Suppose p is a prime number. The possible values of gcd of $p^3 + p^2 + p + 11$ and $p^2 + 1$ are

A) 1,2,5 B) 2,5,10 C) 1,5,10 D) 1,2,10.

Answer: B) Let $a = p^3 + p^2 + p + 11$, $b = p^2 + 1$, and $d =$ gcd of a and b. Then since $a = b(p + 1) + 10$, d divides 10. So $d = 1, 2, 5$ or 10. But for $p = 2$, we get $d = 5$ and for odd p, both a, b are even, hence d is even. So $d \neq 1$.

5. Consider all 2×2 matrices whose entries are distinct and belong to $\{1, 2, 3, 4\}$. The sum of determinants of all such matrices is

A) $4!$ B) 0 C) negative D) odd.

Answer: B) There are in all $4! = 24$ matrices. These can be taken in pairs like $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right)$ where B is obtained by interchanging the rows of A; so $\det A = -\det B$ or $\det A + \det B = 0$.

6. Choose the correct alternative:

A) The Taylor series of $\sin \dfrac{1}{x}$ about $x = \dfrac{2}{\pi}$ does not exist.

B) The coefficient of $\left(x - \dfrac{2}{\pi}\right)^2$ in the Taylor series of $\sin \dfrac{1}{x}$ about $x = \dfrac{2}{\pi}$ is $\dfrac{-\pi^4}{32}$.

C) The Taylor series of $\sin \dfrac{1}{x}$ about $x = \dfrac{2}{\pi}$ has negative powers of x.

D) The coefficient of $\left(x - \dfrac{2}{\pi}\right)^2$ in the Taylor series of $\sin \dfrac{1}{x}$ about $x = \dfrac{2}{\pi}$ is 0.

Answer: B) For $f(x) = \sin(1/x)$, $f'(x) = -(1/x^2)\cos(1/x)$ and $f''(x) = (2/x^3)\cos(1/x) -$

$(1/x^4)\sin(1/x)$. So $f(2/\pi)=1$, $f'(2/\pi)=0$, and $f''(2/\pi)=-\pi^4/16$. So the Taylor series is

$$\begin{aligned} f(x) &= f(2/\pi)+f'(2/\pi)(x-2/\pi)+\frac{1}{2!}f''(2/\pi)(x-2/\pi)^2+\cdots \\ &= 1+\frac{1}{2}\left[\frac{-\pi^4}{16}\right](x-2/\pi)^2+\cdots. \end{aligned}$$

7. Consider all right circular cylinders for which the sum of the height and circumference of the base is 30 cm. The radius of the one with maximum volume is

A) 3 B) 10 C) $\frac{10}{\pi}$ D) $\frac{\pi}{10}$.

Answer: C) Let x and h be the radius of base and height of a cylinder. Then the circumference is $2\pi x+h$. So $2\pi x+h=30$ by data. So the volume is $v(x)=\pi x^2h=\pi x^2(30-2\pi x)=2\pi(15x^2-\pi x^3)$. So $v'(x)=2\pi(30x-3\pi x^2)$, $v''(x)=2\pi(30-6\pi x)$. Now $v'(x)=0$ gives $x=10/\pi$ as the only non-zero critical value of x and $v''(10/\pi)=2\pi(30-60)<0$. So v is maximum at $x=10/\pi$.

8. In how many ways can you express $2^3 3^5 5^7 7^{11}$ as a product of two numbers, ab, where $\gcd(a,b)=1$ and $1<a<b$?

A) 5 B) 6 C) 7 D) 8.

Answer: C) Let $x=2^3$, $y=3^5$, $z=5^7$, $w=7^{11}$. Then the 7 ways are $(x)(yzw)$, $(y)(xzw)$ $(z)(xyw)$, $(w)(xyz)$; $(xy)(zw)$, $(xz)(yw)$, $(yz)(xw)$.

9. The value of $\displaystyle\int_a^b \sin x\,dx$ is

A) $(b-a)\sin c$ B) $(b-a)\cos c$ C) $\frac{\sin c}{b-a}$ D) $\frac{\cos c}{b-a}$

for some real number c such that $a\le c\le b$.

Answer: A) Since the function $f(x)=\sin x$ is continuous in $[a,b]$, by the mean-value theorem there is $c\in[a,b]$ such that the integral $=(b-a)f(c)$.

10. Suppose a,b,c are three distinct integers from 2 to 10 (both inclusive). Exactly one of ab, bc and ca is odd and abc is a multiple of 4. The arithmetic mean of a and b is an integer and so is the arithmetic mean of a, b and c. How many such (unordered) triplets are possible?

A) 4 B) 5 C) 6 D) 7.

Answer: A) Since exactly one of ab, bc and ca is odd and $4|abc$, two of the numbers must be odd and the remaining must be a multiple of 4. Since the A.M., $(a+b)/2$, of a, b is an integer, $2|(a+b)$ so that a, b have the same parity so that both are odd by the above. Since the A.M., $(a+b+c)/3$, of a,b,c is an integer, $3|(a+b+c)$. So the only triplets are $(a,b,c)=(3,5,4)$; $(3,7,8)$; $(5,9,4)$ and $(7,9,8)$.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $P(x)=\sum_{r=0}^{n} c_r x^r$ be a polynomial with real coefficients with $c_0>0$ and $\sum_{r=0}^{[n/2]}\frac{c_{2r}}{2r+1}<0$. Prove that P has root in $(-1,1)$.

Solution : $P(0) = c_0 > 0$. **[1 mk]**

Note that

$$\begin{aligned} I = \int_{-1}^{1} P(x)dx &= \int_{-1}^{0} P(x)dx + \int_{0}^{1} P(x)dx \quad \text{(Put } x = -t) \\ &= \int_{1}^{0} P(-t)(-dt) + \int_{0}^{1} P(x)dx \\ &= \int_{0}^{1} [P(-x) + P(x)]dx \\ &= 2\int_{0}^{1} \sum_{r=0}^{[n/2]} c_{2r}x^{2r}dx = 2\sum_{r=0}^{[n/2]} \frac{c_{2r}}{2r+1}. \end{aligned} \qquad \text{[}\mathbf{3}\text{ mks]}$$

By data, $\displaystyle\sum_{r=0}^{[n/2]} \frac{c_{2r}}{2r+1} < 0$, so that $I < 0$. Now P is a continuous function on $[-1, 1]$ and its integral I over $[-1, 1]$ is *negative*. Hence there is $k \in [-1, 1]$ such that $P(k) < 0$. (Recall: If f is a continuous function on $[a, b]$ $(a < b)$ such that $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq 0$.) Also, $P(0) = c_0 > 0$. Therefore, since P is a continuous function on $[-1, 1]$, there is a number α between k and 0 such that $P(\alpha) = 0$. **[2 mks]**

2. If $|z_1| = |z_2| = |z_3| > 0$ and $z_1 + z_2 + z_3 = 0$, then show that the points representing the complex numbers z_1, z_2, z_3 form an equilateral triangle.
Solution: There are unique numbers $\theta_1, \theta_2, \theta_3$ in $[0, 2\pi)$ such that

$$z_k = r(\cos\theta_k + i\sin\theta_k), \quad k = 1, 2, 3, \qquad (1)$$

where $r = |z_1| = |z_2| = |z_3|$. Then if A, B, C are the points represented by the numbers z_1, z_2, z_3 respectively, then they lie on the circle $|z| = r$ with origin O as its centre. The condition $z_1 + z_2 + z_3 = 0$ gives

$$r[(\cos\theta_1 + \cos\theta_2 + \cos\theta_3) + i(\sin\theta_1 + \sin\theta_2 + \sin\theta_3)] = 0. \qquad \text{[}\mathbf{1}\text{ mk]}$$

As $r > 0$, this gives $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$ and $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0$. Hence

$$\cos\theta_1 + \cos\theta_2 = -\cos\theta_3, \quad \sin\theta_1 + \sin\theta_2 = -\sin\theta_3. \qquad \text{[}\mathbf{2}\text{ mks]}$$

Squaring and adding we get $1 + 1 - 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = 1$ or $\cos(\theta_2 - \theta_1) = -1/2$. So $\angle AOB = \theta_2 - \theta_1 = 2\pi/3$. Similarly $\angle BOC = \angle COA = 2\pi/3$. Hence the chords AB, BC, CA subtend the same angle, $2\pi/3$, at the centre and so $AB = BC = CA$. So $\triangle ABC$ is equilateral. **[3 mks]**

3. If $1, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are n^{th} roots of unity, prove that

$$\frac{1}{2-\alpha_1} + \frac{1}{2-\alpha_2} + \cdots + \frac{1}{2-\alpha_{n-1}} = \frac{(n-2)2^{n-1}+1}{2^n-1}.$$

Solution: First method: Let $f(x) = x^n - 1$ so that $\alpha_0 = 1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the n roots of $f(x) = 0$ and of these $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are non-real for $n \geq 3$. For a fixed r,

$0 \le r \le n-1$, we have the identity

$$\begin{aligned}\frac{x^n-1}{x-\alpha_r} &= \frac{x^n-\alpha_r^n}{x-\alpha_r} \\ &= x^{n-1}+x^{n-2}\alpha_r+x^{n-3}\alpha_r^2+\cdots+x\alpha_r^{n-2}+\alpha_r^{n-1}. \qquad [\mathbf{1}\text{ mk}]\end{aligned}$$

Putting $x=2$ and summing over r we get

$$\begin{aligned}&\sum_{r=0}^{n-1}\frac{2^n-1}{2-\alpha_r} \\ &= \sum_r 2^{n-1}+2^{n-2}\sum_r \alpha_r+\cdots+2\sum_r \alpha_r^{n-2}+\sum_r \alpha_r^{n-1}. \qquad (1)\end{aligned}$$

Now the nth roots of unity are $\alpha_r = e^{(2\pi i)r/n}$, $r=0,1,\ldots,n-1$. Thus $\alpha_r=\alpha_1^r$, $r=0,1,\ldots,n-1$. For a fixed j, $1\le j\le n-1$, we have

$$\sum_{r=0}^{n-1}\alpha_r^j=\sum_{r=0}^{n-1}(\alpha_1^r)^j=\sum_{r=0}^{n-1}(\alpha_1^j)^r=\frac{1-(\alpha_1^j)^n}{1-\alpha_1^j}=0 \qquad [\mathbf{3}\text{ mks}]$$

since $(\alpha_1^j)^n=(\alpha_1^n)^j=1^j=1$. So substituting in (1) we have

$$\begin{aligned}&(2^n-1)\sum_{r=0}^{n-1}\frac{1}{2-\alpha_r}=n2^{n-1} \\ \text{or}\quad &1+\sum_{r=1}^{n-1}\frac{1}{2-\alpha_r}=\frac{n2^{n-1}}{2^n-1}\qquad (\text{as } \alpha_0=1) \\ \text{or}\quad &\sum_{r=1}^{n-1}\frac{1}{2-\alpha_r}=\frac{n2^{n-1}}{2^n-1}-1=\frac{(n-2)2^{n-1}+1}{2^n-1}. \qquad [\mathbf{2}\text{ mks}]\end{aligned}$$

Second method: Let $f(x)=x^n-1$ so that $\alpha_0=1,\alpha_1,\alpha_2,\ldots,\alpha_{n-1}$ are the n roots of $f(x)=0$. Hence we have the identity

$$f(x)=(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1}). \qquad [\mathbf{2}\text{ mks}]$$

Taking logarithm on both the sides and differentiating we get the identity

$$\frac{nx^{n-1}}{x^n-1}=\frac{1}{x-\alpha_0}+\frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\cdots+\frac{1}{x-\alpha_{n-1}},$$

for x different from all the α_i. So putting $x=2$ we get,

$$\begin{aligned}&\frac{1}{2-\alpha_0}+\frac{1}{2-\alpha_1}+\frac{1}{2-\alpha_2}+\cdots+\frac{1}{2-\alpha_{n-1}}=\frac{n2^{n-1}}{2^n-1},\quad (\alpha_0=1) \\ &\frac{1}{2-\alpha_1}+\frac{1}{2-\alpha_2}+\cdots+\frac{1}{2-\alpha_{n-1}}=\frac{n2^{n-1}}{2^n-1}-1=\frac{(n-2)2^{n-1}+1}{2^n-1}. \qquad [\mathbf{4}\text{ mks}]\end{aligned}$$

4. Let $f(x)$ be a monic polynomial of degree 4 such that $f(1) = 10, f(2) = 20, f(3) = 30$. Find $f(12) + f(-8)$.

Solution: First method: Note that $f(1) = 10$ means that the remainder is 10 when $f(x)$ is divided by $x - 1$. Similarly, the remainder is 10×2 when $f(x)$ is divided by $x - 2$ and the remainder is 10×3 when $f(x)$ is divided by $x - 3$. Hence we can take the degree 4 monic polynomial $f(x)$ to be $f(x) = (x-1)(x-2)(x-3)(x-k) + 10x$, where k is a parameter. Then **[4** mks**]**

$$\begin{aligned} f(12) &= 11 \cdot 10 \cdot 9(12 - k) + 120 = 990(12 - k) + 120, \\ f(-8) &= (-9)(-10)(-11)(-8 - k) - 80 = -990(-8 - k) - 80. \end{aligned}$$

Adding, $f(12) + f(-8) = 990(12 - k + 8 + k) + 40 = 990(20) + 40 = 19840$. **[2** mks**]**

Second method: Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ so that by the data we have the equations

$$\begin{aligned} 1 + a + b + c + d &= 10, \\ 16 + 8a + 4b + 2c + d &= 20, \\ 81 + 27a + 9b + 3c + d &= 30. \end{aligned}$$

[2 mks**]**

Reducing this system we get

$$\begin{aligned} a + b + c + d &= 9, \\ 36b + 54c + 63d &= 612, \\ 6c + 11d &= 24. \end{aligned}$$

Solving, we get $c = 4 - \frac{11}{6}d$, $b = 11 + d$, $a = -6 - \frac{1}{6}d$. So **[3 mks]**

$$\begin{aligned} f(x) &= x^4 + (-6 - \frac{1}{6}d)x^3 + (11 + d)x^2 + (4 - \frac{11}{6}d)x + d \\ &= x^4 - 6x^3 + 11x^2 + 4x + d\left[-\frac{1}{6}x^3 + x^2 - \frac{11}{6}x + 1\right]. \end{aligned}$$

Hence $f(12) + f(-8) = 12000 - 165d + 7840 + 165d = 19840$. **[1 mk]**

5. Find all solutions (a, b, c, n) in positve integers for the equation $2^n = a! + b! + c!$.

Solution: Let $a \le b \le c$. Then $N = 1 + \frac{b!}{a!} + \frac{c!}{a!}$ is an integer since $\frac{b!}{a!}$ and $\frac{c!}{a!}$ are both integers. Therefore, since $2^n = a! \cdot N$, we see that $a!$ divides 2^n so that we must have $a = 1$ or $a = 2$.

Let $a = 1$. Then $2^n - 1 = b! + c!$ where left side is odd. So exactly one of $b!$ and $c!$ must be odd. But $k!$ is odd only when $k = 1$. Therefore $b = 1$ since $1 \le b \le c$. So $2^n - 2 = c!$ or $2(2^{n-1} - 1) = c!$ and left side is not divisible by 4. So $c \le 3$, and $(a, b, c, n) = (1, 1, 2, 2), (1, 1, 3, 3)$ are the corresponding solutions. **[4 mks]**

Let $a = 2$. Then $2 \le b \le c$ and $2(2^{n-1} - 1) = b!(1 + \frac{c!}{b!})$. So as before, $b = 2, 3$. Let $b = 2$. Then $2 \le c$ and $4(2^{n-2} - 1) = c!$ which is not possible as 8 divides $c!$ for $c \ge 4$ and 4 does not divide $2!$ and $3!$. Let $b = 3$. Then $3 \le c$ and $8(2^{n-3} - 1) = c!$ so that $4 \le c \le 5$ as 16 divides $c!$ for $c \ge 6$. Here $(a, b, c, n) = (2, 3, 4, 5), (2, 3, 5, 7)$ are the corresponding solutions.

Thus the only possibilities are $2^2 = 1! + 1! + 2!$, $2^3 = 1! + 1! + 3!$, $2^5 = 2! + 3! + 4!$ and $2^7 = 2! + 3! + 5!$. **[2 mks]**

Part III

1. Suppose the polynomials f and g have the same roots and $\{x \in \mathbb{C} : f(x) = 2015\} = \{x \in \mathbb{C} : g(x) = 2015\}$, then show that $f = g$. [13]

Solution: The polynomials f and g have *the same* roots. Let these roots be $\alpha_1, \ldots, \alpha_n$, and let the multiplicities of these be $a_1, \ldots, a_n$ for f and $b_1, \ldots, b_n$ for g respectively. Then the degree of f is $N_1 = a_1 + \cdots + a_n$ and the degree of g is $N_2 = b_1 + \cdots + b_n$, and we have

$$\begin{aligned} f(x) &= (x-\alpha_1)^{a_1}(x-\alpha_2)^{a_2}\cdots(x-\alpha_n)^{a_n}, \\ g(x) &= (x-\alpha_1)^{b_1}(x-\alpha_2)^{b_2}\cdots(x-\alpha_n)^{b_n}, \\ \text{where} \quad & N_1 \geq N_2, \quad \text{say.} \end{aligned} \tag{1}$$

[**2** mks]

Next, by data the polynomials $F(x) = f(x) - 2015$ and $G(x) = g(x) - 2015$ also have *the same* roots, say $\beta_1, \ldots, \beta_m$. Let their multiplicities be $a'_1, \ldots, a'_m$ for F and $b'_1, \ldots, b'_m$ for G respectively. Then degree of F = degree of f = $N_1 = a'_1 + \cdots + a'_m$ and degree of G = degree of g = $N_2 = b'_1 + \cdots + b'_m$, and we have

$$\begin{aligned} F(x) &= (x-\beta_1)^{a'_1}(x-\beta_2)^{a'_2}\cdots(x-\beta_m)^{a'_m}, \\ G(x) &= (x-\beta_1)^{b'_1}(x-\beta_2)^{b'_2}\cdots(x-\beta_m)^{b'_m}. \end{aligned}$$

Note that the sets $S = \{\alpha_1, \ldots, \alpha_n\}$ and $T = \{\beta_1, \ldots, \beta_m\}$ are *disjoint.*

Let, if possible, $f \neq g$ i.e. let $f - g$ be a *non-zero* polynomial. Then as $\alpha_1, \ldots, \alpha_n$ are roots of both f and g, they are roots of $f - g$ also. Similarly, $\beta_1, \ldots, \beta_m$ are roots of $F - G = f - g$. Therefore, since the sets S, T are disjoint, we have

$$f - g = (x-\alpha_1)\cdots(x-\alpha_n)(x-\beta_1)\cdots(x-\beta_m)H(x), \qquad [\mathbf{5}\text{ mks}]$$

for some polynomial $H(x)$. Hence $f - g$ is a *non-constant* polynomial of degree say K, and

$$K \geq n + m. \tag{2}$$

To obtain a contradiction we consider the multiplicities of the roots of the derivative f'. Now α_1 is a root of f with multiplicity a_1 means that

$$f(x) = (x-\alpha_1)^{a_1}\phi(x)$$

where $\phi(\alpha_1) \neq 0$. This gives $f'(x) = a_1(x-\alpha_1)^{a_1-1}\phi(x) + (x-\alpha_1)^{a_1}\phi'(x)$. So $f'(x) = (x-\alpha_1)^{a_1-1}\psi(x)$ where $\psi(x) = a_1\phi(x) + (x-\alpha_1)\phi'(x)$. Therefore, since $\psi(\alpha_1) = a_1\phi(\alpha_1) \neq 0$, we see that α_1 is a root of f' with multiplicity $a_1 - 1$. Thus $\alpha_1, \ldots, \alpha_n$ are roots of f' with multiplicities $a_1 - 1, \ldots, a_n - 1$. Similarly, since $f' = F'$, it follows that $\beta_1, \ldots, \beta_m$ are also roots of f' with multiplicities $a'_1 - 1, \ldots, a'_m - 1$. Hence the degree of f' namely, $N_1 - 1$, is such that

$$\begin{aligned} N_1 - 1 &\geq \sum_{i=1}^{n}(a_i - 1) + \sum_{j=1}^{m}(a'_j - 1) = N_1 - n + N_1 - m \\ \text{or} \quad n + m - 1 &\geq N_1. \end{aligned} \tag{3}$$

By (1), the degree K of $f-g$ satisfies $K \le N_1$. So by (2) and (3), we get $n+m \le K \le N_1 \le n+m-1$ or $n+m \le n+m-1$, contradiction. So $f = g$. **[6 mks]**

2. Give an example of a function which is continuous at exactly two points and differentiable at exactly one of them. Justify your answer. [13]

Solution Define the function $f : \mathbb{R} \to \mathbb{R}$ thus:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ x^3 & \text{if } x \text{ is irrational} \end{cases} \qquad \textbf{[5 mks]}$$

We show that f is continuous only at 0 and 1, and differentiable only at 0. For this, consider a real number a. Then as $x \to a$ through rational values, $f(x) = x^2 \to a^2$, and as $x \to a$ through irrational values, $f(x) = x^3 \to a^3$. So the limit $\lim_{x\to a} f(x)$ will exist if and only if the above two limits are equal i.e. if and only if $a^2 = a^3$ i.e. $a^2(a-1) = 0$ i.e. $a = 0$ or $a = 1$. Thus f is continuous at 0 since $\lim f(x) = \lim x^2 = 0 = f(0)$. Similarly, f is continuous at 1. But when $a \ne 0, 1$, $\lim_{x\to a} f(x)$ does not exist; so f is discontinuous at a. **[4 mks]**

Next, let $g(x) = [f(x) - f(a)]/(x-a)$. Let a be rational. As $x \to a$ through irrational values, $\lim g(x) = \lim\{[x^3 - a^2]/(x-a)\}$ is not finite if $\lim[x^3 - a^2] \ne 0$ i.e. if $a^3 \ne a^2$ i.e. if $a \notin \{0, 1\}$. Hence $f'(a)$ does not exist (finitely) if $a \notin \{0, 1\}$. Let $a = 0$. Then $\lim_{x\to 0} g(x) = \lim_{x\to 0}[f(x)/x] = 0$. So $f'(0)$ exists and is 0.

But as $x \to 1$ through rational values, $\lim g(x) = \dfrac{x^2-1}{x-1} = 2$, while as $x \to 1$ through irrational values, $\lim g(x) = \lim \dfrac{x^3-1}{x-1} = 3$. Hence $f'(1)$ does not exist.

Let a be irrational. As $x \to a$ through rational values,

$$\lim g(x) = \lim\{[x^2 - a^3]/(x-a)\}$$

is not finite if $\lim[x^2 - a^3] \ne 0$ i.e. if $a^2 \ne a^3$ i.e. if $a \notin \{0, 1\}$. Hence $f'(a)$ does not exist. **[4 mks]**

3. Let A be any $m \times n$ matrix whose entries are positive inegers. A step consists of transforming the matrix either by multiplying every entry of *a row* by 2 or subtracting 1 from every entry of *a column*. Can you transform A into the zero matrix in finitely many steps? Justify your answer. [12]

Solution Yes, one method is as follows: Let $A = [a_{ij}]$ be the matrix. Let m be the minimum element in the first column C_1. In fact, let m occur s times i.e. let $m = a_{i_1 1} = \cdots = a_{i_s 1}$. We may *assume* that $m = 1$. For if $m \ge 2$, subtract 1 from each element of C_1 $m-1$ times so that the minimum element in C_1 is 1. **[4 mks]**

Multiply each of the s rows $i_1, i_2, \ldots, i_s$ of A by 2. This forces the minimum element in C_1 to be 2. Subtract 1 from each element of C_1. The effect of these steps on C_1 is this: the s elements $a_{i_1 1}, a_{i_2 1}, \ldots, a_{i_s 1}$ of C_1 are still equal to 1, but the remaining elements of C_1 have all become *smaller* though they are all still ≥ 1. Hence in a finite number of steps *all* elements of C_1 will become 1. Then subtracting 1 from each element of C_1 makes C_1 a column of zeros.

Next make the second column C_2 a column of zeros as in the above. Note that the operations on C_2 have no effect on C_1 and C_1 *remains a column of zeros.* Hence in a finite number of steps A becomes the zero matrix. **[8 mks]**

4. Let S be the set of positive integers that do not have zero in their decimal representation. Thus $S = \{1, 2, 3, \cdots, 9, 11, 12, \cdots, 19, 21, \cdots, 99, 111, \cdots\}$.
   Show that the series $\sum_{n \in S} \frac{1}{n}$ converges. [12]

Solution: Let A denote the series $\sum_{n \in S} \frac{1}{n}$. Let t_1 denote the sum of the first 9 terms of A, namely $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{9}$. Then $t_1 < 9$ since each of these 9 terms is ≤ 1. **[2 mks]**

Next, let t_2 denote the sum of the *next* $81 = 9^2$ terms of A, namely

$$\begin{array}{l} \frac{1}{11}, \frac{1}{12}, \ldots, \frac{1}{19}, \\ \frac{1}{21}, \frac{1}{22}, \ldots, \frac{1}{29}, \\ \cdots \quad \cdots \quad \cdots \\ \frac{1}{91}, \frac{1}{92}, \ldots, \frac{1}{99}. \end{array}$$

[**3** mks]

Then $t_2 < \frac{1}{10} \cdot 9^2$ since each of these 9^2 terms is $\leq \frac{1}{11} < \frac{1}{10}$.

Similarly, let t_3 denote the sum of the *next* $729 = 9^3$ terms of A, namely

$$\begin{array}{l} \frac{1}{111}, \frac{1}{112}, \ldots, \frac{1}{119}, \\ \frac{1}{221}, \frac{1}{222}, \ldots, \frac{1}{229}, \\ \cdots \quad \cdots \quad \cdots \\ \frac{1}{991}, \frac{1}{992}, \ldots, \frac{1}{999}. \end{array}$$

[**3** mks]

Then $t_3 < \frac{1}{10^2} \cdot 9^3$ since each of these 9^3 terms is $\leq \frac{1}{111} < \frac{1}{100}$.

Proceeding in this way, if s_n is the sum of the first n terms of A, we take m to be the smallest positive integer such that $n \leq 9 + 9^2 + \cdots + 9^m = 9(9^m - 1)/8 = N$, say. Then since the terms are positive and since the first n terms of A are included in the first N terms of A, we see that

$$\begin{aligned} s_n < s_N &= 9 + \frac{1}{10} \cdot 9^2 + \frac{1}{10^2} \cdot 9^3 + \cdots + \frac{1}{10^{m-1}} \cdot 9^m \\ &= 9 \left[1 + \frac{9}{10} + \left(\frac{9}{10} \right)^2 + \cdots + \left(\frac{9}{10} \right)^{m-1} \right] \\ &= 9 \cdot \frac{1 - (9/10)^m}{1 - (9/10)} < 90. \end{aligned}$$

So the partial sums of the positive term series A are bounded above. Hence series A is convergent. **[4 mks]**